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A solution of an operator equation related to the KdV equation

Ramesh V. Garimella [a], Volodymyr Hrynkiv [b,*], A.R. Sourour [c]

[a] *Department of Mathematics, University of Central Arkansas, Conway, AR 72035, USA*
[b] *Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA*
[c] *Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada V8W 3P4*

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Abstract

For a given nonzero bounded linear operator A on a Banach space X, we show that if A or A^* has an eigenvalue then, except when the dimension of X is equal to two and the trace of A is zero, there exists a bounded linear operator B on X such that (i) $AB + BA$ is of rank one, and (ii) $I + f(A)B$ is invertible for every function f analytic in a neighborhood of the spectrum of A. This result was motivated by the operator method used by Carl et al. [H. Aden, B. Carl, On realizations of solutions of the KdV equation by determinants on operator ideals, J. Math. Phys. 37 (1996) 1833–1857; H. Blohm, Solution of nonlinear equations by trace methods, Nonlinearity 13 (2000) 1925–1964; B. Carl, C. Schiebold, Nonlinear equations in soliton physics and operator ideals, Nonlinearity 12 (1999) 333–364; B. Carl, S.-Z. Huang, On realizations of solutions of the KdV equation by the C_0-semigroup method, Amer. J. Math. 122 (2000) 403–438; S.-Z. Huang, An operator method for finding exact solutions to vector Korteweg–de Vries equations, J. Math. Phys. 44 (2003) 1357–1388] to solve nonlinear partial differential equations such as the Korteweg–deVries (KdV), modified KdV, and Kadomtsev–Petviashvili equations.

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* Corresponding author.
E-mail addresses: rameshg@uca.edu (R.V. Garimella), hrynkiv@math.utk.edu (V. Hrynkiv), sourour@math.uvic.ca (A.R. Sourour).

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1. Introduction

In [1], Aden and Carl used a method known as operator method to find solutions to the scalar KdV equation $v_t = v_{xxx} + 3v_x^2$. A similar method was also used by Blohm in [2], Carl and Schiebold in [3], Carl and Huang in [4], and Huang in [5] to solve some other PDEs such as the modified KdV equation, Kadomtsev–Petviashvili equation, and the sine-Gordon equation. The main idea behind the operator method can be described as follows. Given a nonlinear PDE and a specific scalar solution to the equation, the first step in finding other solutions is to translate the given nonlinear equation to an operator equation. Using the specific scalar solution as an aid, one then searches for a family of operator solutions to the operator equation. Having obtained the operator solutions, the second step is to transfer the operator-valued solutions into scalar solutions by using a suitable scalarization technique. The first step is the most important one in the above operator method. In order to accomplish this step, one often needs to know the following: given an operator A on the complex sequence space ℓ^2 consisting of square summable sequences, is it possible to find an operator B on ℓ^2 such that

1. $AB + BA$ is of rank one, and
2. the operator $I + L$ is invertible, where $L := \mathrm{e}^{p(A)}B$, for a polynomial p. (The expression of L usually appears in the literature in the form $\mathrm{e}^{xP(A)+tQ(A)}B$ for polynomials P and Q and real numbers x and t.)

In this paper, we investigate the above question for a bounded linear operator on a Banach space. As a matter of fact, if a bounded linear operator A on a Banach space X (except when the dimension of X is two and the trace A is zero) has an eigenvalue, then there exists a bounded linear operator B on X such that (i) $AB + BA$ is of rank one, and (ii) $I + f(A)B$ is invertible for every function f analytic in a neighborhood of the spectrum of A.

We now fix some notation and terminology. The linear span of a subset S of a vector space is denotes by span(S). If T is a bounded operator on a Banach space, then T^* denotes the adjoint of T. That is, T^* is the linear operator defined on the dual space X' by $(T^*\phi)(x) = \phi(Tx)$ for each $x \in X$ and $\phi \in X'$. A linear operator $T : E \to F$, where E and F are Banach spaces, is said to be of rank one if the dimension of the range of T is one. It is straightforward to verify that a bounded operator T is of rank one if and only if there exists $\phi \in E'$ (dual of E) and $y \in F$ such that $T = \phi \otimes y$, where $(\phi \otimes y)x := \phi(x)y$, for every $x \in E$. It is obvious that the map $(\phi, x) \mapsto \phi \otimes x$ is a bilinear map. It is also easy to verify that for every $x, y \in E$; $\phi, \psi \in E'$ and a bounded operator T on E, we have

(a) $T \cdot \phi \otimes x = \phi \otimes Tx$,
(b) $\phi \otimes x \cdot T = T^*\phi \otimes x$, in particular,
(c) $(\phi \otimes x) \cdot (\psi \otimes y) = \psi \otimes \phi(y)x$.

2. Main results

Theorem 2.1. *Let A be a nonzero bounded linear operator on a Banach space X, where* $\dim(X) \geqslant 3$. *If the point spectrum of A or of* A^* *is nonempty, then there exists a bounded linear operator B such that*

(i) *$AB + BA$ is of rank one, and*
(ii) *$I + f(A)B$ is invertible for every function f, which is analytic in a neighborhood of the spectrum of A.*

Remark 1. If X is a finite dimensional space then the spectrum of any linear operator on X is precisely the point spectrum and is obviously nonempty.

Remark 2. The second condition in the above theorem is equivalent to $f(A)B$ being a quasinilpotent operator on X.

Proof. First, assume that the point spectrum of A is nonempty. Let v be an eigenvector of A corresponding to an eigenvalue λ. We divide the proof into two cases.

Case I. Suppose that $\text{Range}(A + \lambda I) \nsubseteq \text{span}\{v\}$. By the Hahn–Banach Theorem there exists a bounded linear functional ϕ on X such that $\phi(v) = 0$ but $\phi(\text{Range}(A + \lambda I)) \neq \{0\}$. The latter inequality implies that $(A^* + \lambda I)\phi \neq 0$ (by [6, Theorem 4.12] or [7, III; Eq. (3.2)]). We claim that $B := \phi \otimes v$ satisfies conditions (i) and (ii).

Indeed,

$$\begin{aligned} AB + BA &= A(\phi \otimes v) + (\phi \otimes v)A \\ &= \phi \otimes Av + A^*\phi \otimes v \\ &= \phi \otimes \lambda v + A^*\phi \otimes v \\ &= (A^* + \lambda I)\phi \otimes v. \end{aligned}$$

This proves that $AB + BA$ is of rank one.

Let f be an analytic function on an open set containing the spectrum of A. It follows that $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda)}{n!}(z - \lambda)^n$ for every z in a neighborhood of λ. Then

$$f(A)v = \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda)}{n!}(A - \lambda I)^n(v) = f(\lambda)v$$

and

$$\begin{aligned} f(A)B &= f(A)(\phi \otimes v) \\ &= \phi \otimes f(A)v \\ &= \phi \otimes f(\lambda)v. \end{aligned}$$

Since $\phi(v) = 0$, it is obvious that $(f(A)B)^2 = 0$. Hence $I + f(A)B$ is invertible. This proves the claim.

Case II. $\text{Range}(A + \lambda I) \subseteq \text{span}\{v\}$. We consider two subcases according to whether λ is zero or not.

If $\lambda \neq 0$, we note that $\text{Range}(A + \lambda I) \neq \{0\}$ since $(A + \lambda I)v = 2\lambda v$. Therefore $\text{Range}(A + \lambda I) = \text{span}\{v\}$, i.e., $A + \lambda I$ has rank one, hence $\text{Null}(A + \lambda I)$ is a subspace of codimension 1. This implies that not only is $-\lambda$ an eigenvalue of A but the corresponding eigenspace has dimension at least two. We choose an eigenvector w corresponding to the eigenvalue $-\lambda$. Since $\text{Null}(A + \lambda I) \subseteq \text{Range}(A - \lambda I)$, we have that $\dim(\text{Range}(A - \lambda I)) \geqslant 2$, and so $\text{Range}(A - \lambda I) \nsubseteq \text{span}\{w\}$. The proof now proceeds as in Case I with $B = \psi \otimes w$, for a linear functional ψ vanishing on w but not on $\text{Range}(A - \lambda I)$.

If $\lambda = 0$, then we have $\text{Range}(A) \subseteq \text{span}\{v\}$ and since $A \neq 0$, we have $\text{Range}(A) = \text{span}\{v\}$. Thus $A = \phi \otimes v$ for some $\phi \in X'$. Since $Av = 0$, we have $\phi(v) = 0$. Since $\dim X \geqslant 3$, we have that $\dim(\text{Null}(\phi)) \geqslant 2$. Let x be a vector in $\text{Null}(\phi)$ that is linearly independent of v. By the Hahn–Banach Theorem, there exists a linear functional $\psi \in X'$ such that $\psi(v) = 1$ and $\psi(x) = 0$. We will show that $B := \psi \otimes x$ satisfies conditions (i) and (ii).

We have $AB = 0$ since $\phi(x) = 0$ and $BA = \phi \otimes x$ since $\psi(v) = 1$. Therefore $AB + BA = \phi \otimes x$, a rank one operator.

Assume that f is analytic in a neighborhood of $\{0\}$, the spectrum of A. Since $A^2 = 0$, we have $f(A) = f(0)I + f'(0)A$ and hence $f(A)B = f(0)B + f'(0)AB = f(0)B$ which is nilpotent. Thus $I + f(A)B$ is invertible. This ends the proof under the assumption that the point spectrum of A is nonempty.

Now assume that the point spectrum of A^* is nonempty and take an eigenvector $\phi \in X'$ corresponding to an eigenvalue λ. We shall make use of the weak* topology on X'. Recall that this is a locally convex topology and that every weak*-continuous linear functional on X' is of the form $\psi \mapsto \psi(x)$ for some $x \in X$ (see [6, Chapter 4]). We denote this linear functional on X' by $\hat{x}$. If we assume, as in case I above, that $\text{Range}(A^* + \lambda I) \nsubseteq \text{span}\{\phi\}$ then we may apply the Hahn–Banach Theorem for locally convex spaces ([6, Theorem 3.4]) to find a vector $x \in X$ such that $\hat{x}(\phi) = \phi(x) = 0$ but $\hat{x}(\text{Range}(A^* + \lambda I)) \neq 0$. Again, by ([6, Theorem 4.12]) or ([7, III; Eq. (3.3)]), we have $(A + \lambda I)x \neq 0$. We take $B = \psi \otimes x$. The proof now proceeds exactly as before.

When $\text{Range}(A^* + \lambda I) \subseteq \text{span}\{\phi\}$, then as in case (II) above, we may either replace λ by $-\lambda$ reducing to case (I) again or conclude that A^* has rank one. In the latter case, it is well-known, that A itself is of rank one; hence the conclusion follows from the first part of the proof. □

Remark 3. The proof for the case when A is rank one nilpotent may be illustrated by the following matrix construction. Let $n \geqslant 3$ and let A be an $n \times n$ matrix which is nilpotent of rank one. Without loss of generality, we may assume that A has 1 in the first row and second column and zeros elsewhere. Let B be the $n \times n$ matrix which has 1 in the last row and first column and zeros elsewhere. Clearly $AB = 0$, $B^2 = 0$, and BA is of rank one. If f is analytic in a neighborhood of zero, then $f(A)B = (f(0)I + f'(0)A)B = f(0)B$ which is nilpotent. Hence $I + f(A)B$ is invertible.

Next we consider the case when X is of dimension 2. We will show that, when the trace of A is nonzero, there exists an operator B satisfying the conclusion of Theorem 2.1. When A has zero trace, we show that no matrix B satisfies conditions (i) and (ii) even if in condition (ii) we consider only the restricted class of analytic functions (exponentials of polynomials) that appear in solutions to the KdV equation, i.e., condition (2) in Section 1. As this class of functions is invariant under multiplications by nonzero complex numbers, we conclude as before that condition (2) is equivalent to the quasi-nilpotentce of $e^{p(A)}B$.

Theorem 2.2. *Let A be a nonzero* 2×2 *complex matrix. Then there exists a matrix B satisfying conditions* (i) *and* (ii) *of Theorem* 2.1 *if and only if* $\text{trace}(A) \neq 0$.

Proof. First assume that $\text{trace}(A) \neq 0$. Then A has a nonzero eigenvalue λ. Let v be a corresponding eigenvector. Since the trace of A is not zero, $-\lambda$ is not an eigenvalue, so $A + \lambda I$ is surjective and so $\text{Range}(A + \lambda I) \nsubseteq \text{span}\{v\}$. The proof now proceeds as the proof of Theorem 2.1 (Case I).

Next assume that trace$(A) = 0$. We consider two cases according to whether A has nonzero eigenvalues or not.

If zero is the only eigenvalue of A, then without loss of generality, we may assume that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Let f be a function analytic in a neighborhood of zero. Since $A^2 = 0$ it follows that $f(A) = f(0)I + f'(0)A$. In particular, $e^{cA} = I + cA = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ for every $c \in \mathbb{C}$. If $B = [b_{ij}]$ satisfies condition (ii) for these exponentials, then $e^{cA}B$ is nilpotent for every c. But

$$e^{cA}B = \begin{bmatrix} b_{11} + cb_{21} & b_{12} + cb_{22} \\ b_{21} & b_{22} \end{bmatrix}.$$

This matrix must then have zero trace and determinant for every complex number c. It is straightforward to conclude that $b_{11} = b_{21} = b_{22} = 0$. But then we would have $AB = BA = 0$ contradicting condition (i).

If A has a nonzero eigenvalue and zero trace, then we may assume, with no loss of generality, that $A = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$ where $\lambda \neq 0$. For every nonzero complex numbers α and γ, there exists a polynomial p such that $D_{\alpha\gamma} := \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix} = e^{p(A)}$. Indeed using Lagrange interpolation polynomials, one can find a polynomial p that satisfies $p(\lambda) = \log \alpha$ and $p(-\lambda) = \log \gamma$ where log is a branch of the logarithm. If condition (ii) is satisfied for a matrix $B = [b_{ij}]$ and every f which is an exponential of a polynomial, then we must have that $D_{\alpha\gamma}B$ nilpotent. That is $\begin{bmatrix} \alpha b_{11} & \alpha b_{12} \\ \gamma b_{21} & \gamma b_{22} \end{bmatrix}$ has zero trace and determinant for every nonzero α and γ. This easily implies that $b_{11} = b_{22} = 0$ and that b_{12} or b_{21} is also zero. But then $AB + BA = 0$ contradicting condition (i). $\square$

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