Work on the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then, attempt the tiebreaker questions at the end starting with Tie Breaker #1, then #2, and finally #3. Turn in your answer sheet and the tiebreaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Figures aren't necessarily drawn to scale. Angles are given in radians unless otherwise stated. Assume all values are real.

1. Consider the function $f(x) = 5x^{2/3} - 3x$ on the interval $[-1, 8]$. Which of the following is true?
   A. There exists values $c, d \in [-1, 8]$ such that $f(c) \le f(x) \le f(d)$ for all $x \in [-1, 8]$.
   B. There exists a $c \in [-1, 8]$ such that $f(c) = \frac{1}{9}\int_{-1}^{8} f(x)\,dx$.
   C. There exists a $c \in (-1, 8)$ such that $f'(c) = -\frac{4}{3}$.
   D. Both A & B
   E. Both B & C

2. Find the area bounded between the function $f(x) = \frac{\sin(x)}{x} + x$ and $g(x) = x$ on the interval $(-\pi, 2\pi)$.
   A. 0.055
   B. 3.270
   C. 3.704
   D. 4.138
   E. None of the above

3. Let $f(x) = ax^2 - acx + bx - bc$ and $g(x) = x - c$. What would be true about the graph of $y = \frac{f(x)}{g(x)}$?
   A. The graph has a vertical asymptote at $x = c$
   B. The graph has a slant asymptote at $y = ax + b$
   C. The graph has a hole at $(c, a(c) + b)$
   D. A and B
   E. B and C

4. Given $f(x) = \frac{(x-e)\cdot\ln x}{x-e}$, and $\lim_{x\to e} f(x) = 1$. By the definition of a limit there exists a positive real number δ such that $|f(x) - 1| < 0.075$ for each x where $0 < |x - e| < \delta$. Which of the following below is largest valid value of δ that meets the definition, rounded two the nearest hundredth?
   A. 0.19
   B. 0.20
   C. 0.21
   D. 0.22
   E. 0.24

5. Given that the velocity of an object is given by $v(t) = t^3 + 3t^2 - t$, what is the average acceleration on the interval $[-2, 2]$?
   A. 3
   B. 4
   C. 11
   D. 12
   E. 16

6. Evaluate $\int x \sec^2 x \ dx$.
   A. $\frac{x^2}{2}\tan x + C$
   B. $\sec^2 x + 2\sec^2 x \tan x + C$
   C. $x \tan x - \ln|\cos x| + C$
   D. $x \tan x + \ln|\cos x| + C$
   E. None of the above

7. Let $G'(x) = \frac{x + x\cos(x^2)}{\ln(x)}$. If $G(10) = 5$, then $G(3) =$
   A. -24.089
   B. -19.089
   C. 19.089
   D. 24.089
   E. 29.089

8. Evaluate $\lim_{x \to 0} \left(\frac{4x^2}{5 - 5\cos(2x)} \right) =$

A. $\frac{1}{5}$

B. $\frac{2}{5}$

C. $-\frac{8}{5}$

D. $\frac{8}{5}$

E. The limit does not exist

9. Suppose the position of an object is given by $s(t) = t^4 - 6t^2 + 8t - 10$. On what interval(s) of time is the speed increasing?

A. $(-\infty, 0)$

B. $(-\infty, -2)$

C. $(-\infty, -1) \cup (1, \infty)$

D. $(-2, -1) \cup (1, \infty)$

E. None of the above

10. The area bounded by $f(x) = x^2 - 1$ and $g(x) = x + 5$ is a base of a fixed solid such that each cross section cut perpendicular to the x-axis is a semicircle. What is the volume of the solid?

A. 10.227

B. 16.362

C. 40.906

D. 52.083

E. None of the above

11. The graph of the derivative of function f is shown in the figure to the right. The graph has zeros at $x = -6, -3, 1,$ and 5. The graph has horizontal tangent lines at $x = -5, -1,$ and 3.5.
At which of the following values of x does f change from concave down to concave up?

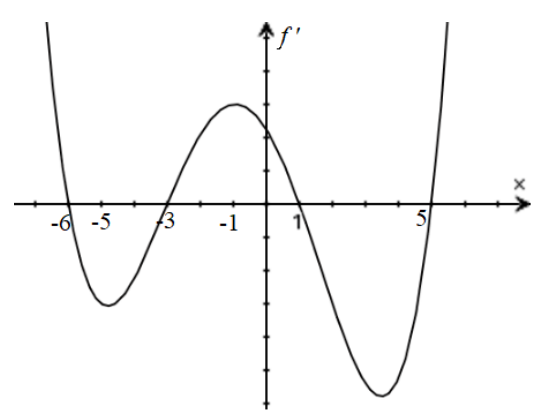
A. $x = -6$ and $x = 1$

B. $x = -5$ and $x = 3.5$

C. $x = -3$ and $x = 5$

D. $x = -1$

E. $x = -6$ and $x = 1$



12. Given $f(x) = x^3 - 3x^2$, find the value(s) of c guaranteed by the Mean Value Theorem on the interval $[-2, 4]$.
   A. $c = 0, 2$
   B. $c = 0, 3$
   C. $c = 1 \pm \sqrt{5}$
   D. $c = 1 \pm \sqrt{3}$
   E. None: MVT does not apply on this interval.

13. The velocity of an object on the x-axis is given by $v(t) = t^3 - 4t + 3$ at t seconds. In this situation, what does $\int_{-3}^{2} v(t)dt$ represent?
   A. Average acceleration from $t = -3$ seconds to $t = 2$ seconds.
   B. Average velocity from $t = -3$ seconds to $t = 2$ seconds.
   C. The distance travelled from $t = -3$ seconds to $t = 2$ seconds.
   D. The change in the position from $t = -3$ seconds to $t = 2$ seconds.
   E. None of the above

14. If $\frac{dy}{dx} = y\cos(x)$ and $y = 5$ when $x = 0$, then $y =$
   A. $e^{\sin(x)} + 4$
   B. $e^{\sin(x)} + 5$
   C. $5e^{\sin(x)}$
   D. $\sin(x) + 5$
   E. $\sin(x) + 5e^x$

15. Determine the slope of $f(x) = \ln(x^2 + 4 + e^{-3x})$ when $x = 0$.
   A. 0
   B. $\frac{1}{5}$
   C. $-\frac{1}{5}$
   D. $\frac{3}{5}$
   E. $-\frac{3}{5}$

16. Find the area inside one petal of the rose defined by the polar equation $r(\theta) = 5\cos(4\theta)$ where θ is measured in radians.

A. 2.5
B. 4.909
C. 9.817
D. 19.635
E. None of the above

17. Consider $f(x) = \begin{cases} x & \text{for } x \leq 1 \\ \frac{1}{x} & \text{for } x > 1 \end{cases}$. Evaluate $\int_0^e f(x)dx =$

A. 0
B. $\frac{3}{2}$
C. 2
D. $\frac{1}{2} + e$
E. $\frac{1}{2} + \ln(e^2)$

18. Let f be the function with derivative given by $f'(x) = x^2 + \frac{2}{x}$. On which of the following intervals is the function f decreasing?

A. $(-\infty, -\sqrt[3]{2})$
B. $(-\sqrt[3]{2}, \infty)$
C. $(-\sqrt[3]{2}, 0)$
D. $(0, \infty)$
E. $(-1, \infty)$

19. Find $\int \frac{1}{\sqrt{-9x^2+9}} dx$.

A. $-\frac{1}{3}\ln\left(\sqrt{x^2-1} + x\right) + C$
B. $\frac{1}{9}\sqrt{-x^2+9} + C$
C. $-\frac{1}{6}\sqrt{x^2-9} + C$
D. $\frac{1}{3}\arcsin(x) + C$
E. None of the above.

20. Consider the differential equation $\frac{dy}{dx} = 2x^2y + 3$. Find $\frac{d^2y}{dx^2}$ in terms of x and y.

A. $\frac{d^2y}{dx^2} = 4xy + 2x^2$

B. $\frac{d^2y}{dx^2} = 4x^2y + 6x^2$

C. $\frac{d^2y}{dx^2} = 4xy + 4x^4y + 6x^2$

D. $\frac{d^2y}{dx^2} = 8x^3y + 12x$

E. None of the above

21. Evaluate $\int_3^\infty \frac{4}{(x-2)^2}\,dx =$

A. 0

B. 2

C. $\int_2^3 \frac{4}{(x-2)^2}\,dx$

D. 4

E. ∞

22. Evaluate the following expression:

$$\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^2 =$$

A. $\int_2^3 x^2\,dx$

B. $\int_2^5 x^2\,dx$

C. $\int_2^3 (2+x)^2\,dx$

D. $\int_2^5 (2+x)^2\,dx$

E. None of the above

23. Evaluate $\int \frac{1}{(x-1)(x+3)}\,dx =$

A. $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$

B. $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right| + C$

C. $\frac{1}{2}\ln|(x-1)(x+3)| + C$

D. $\frac{1}{2}\ln\left|\frac{2x+2}{(x-1)(x+3)}\right| + C$

E. $\ln|(x-1)(x+3)| + C$

24. The table beside gives the velocity, $v(t)$, in miles per hour, of a car at selected time, t, in hours. Using a trapezoidal sum with the three subintervals indicated by the table, what is the approximate distance, in miles, the truck traveled from $t = 0$ to $t = 3$?

t	0	0.5	2	3
$v(t)$	20	60	40	30

A. 140

B. 130

C. 125

D. 120

E. None of the above.

25. Find $\int \frac{x\,e^{x^2}}{7+e^{(x^2)}}\,dx$.

A. $\frac{1}{2\sqrt{7}}\ \arctan\left(\frac{e^{x^2}}{\sqrt{7}}\right) + C$

B. $2\ln\left|7 + e^{x^2}\right| + C$

C. $2\ln(7) + 2x^2 + C$

D. $\frac{1}{2}\ln\left|7 + e^{x^2}\right| + C$

E. None of the above.

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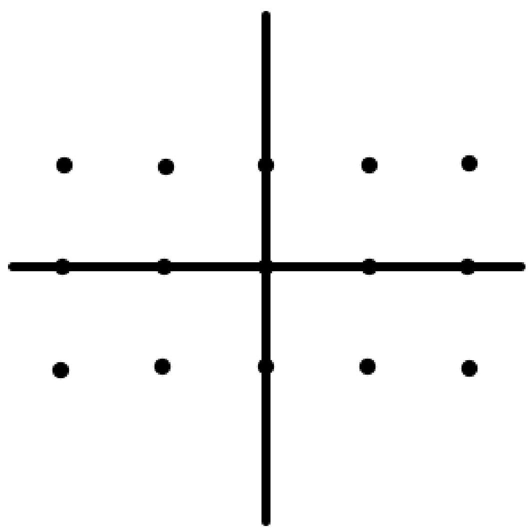
Tie Breaker #1

Name: ___________________

School: __________________

Consider the differential equation $\frac{dy}{dx} = \frac{1}{x-1}$.

A. Sketch a slope field for the differential equation at the fifteen points indicated.



B. Find a particular solution $y = f(x)$ to the differential equation with the initial condition $(-1,0)$ and state its domain.

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Tie Breaker #2

Name: ___________________

School: __________________

Show all work and provide exact values of solutions.

For $0 \leq t \leq 6$ seconds, a screen saver on a computer screen shows two circles that start as dots and expand outward.

a) At the instant that the first circle has a radius of 9 centimeters, the radius is increasing at a rate of $\frac{3}{2}$ centimeters per second. Find the rate at which the area of the circle is changing at that instant. Indicate units of measure.

b) The radius of the first circle is modeled by $w(t) = 12 - 12e^{-0.5t}$ for $0 \leq t \leq 6$, where $w(t)$ is measured in centimeters and t is measured in seconds. At what time t is the radius of the circle increasing at a rate of 3 centimeters per second?

c) A model for the radius of the second circle is given by the function f for $0 \leq t \leq 6$, where $f(t)$ is measured in centimeters and t is measured in seconds. The rate of change of the radius of the second circle is given by $f'(t) = t^2 - 4t + 4$. Based on this model, by how many centimeters does the radius of the second circle increase from time $t = 0$ to $t = 3$?

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Tie Breaker #3

Name: ___________________

School: __________________

Consider the function $f(x) = \ln(x) + 1$.

a) What point(s) on the graph of $f(x) = \ln(x) + 1$ is closest to the coordinate $(3, 0)$?
Round your coordinates to the nearest thousandth.

b) Find the volume created if the area bounded by $f(x) = \ln(x) + 1$, $x = 3$, and $y = 0$ is rotated around the y-axis.
Write the integral used to find your answers. Round your answer to the nearest thousandth.

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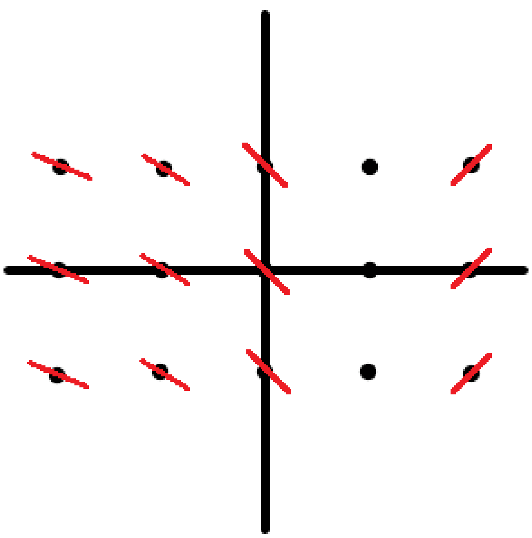
Solutions to 2024 State Calculus Exam

1	D	11	B	21	D
2	D	12	D	22	B
3	C	13	D	23	A
4	A	14	C	24	B
5	A	15	E	25	D
6	D	16	B		
7	B	17	B		
8	B	18	C		
9	D	19	D		
10	C	20	C		

Tie Breaker 1 Answer

Consider the differential equation $\frac{dy}{dx} = \frac{1}{x-1}$.

A. Sketch a slope field for the differential equation at the fifteen points indicated.



$\frac{dy}{dx}$ is explicitly in terms of x and there the slope at a given x-value will all be the same.

x	$\frac{dy}{dx}$
-2	$\frac{1}{-2-1} = -\frac{1}{3}$
-1	$\frac{1}{-1-1} = -\frac{1}{32}$
0	$\frac{1}{0-1} = -1$
1	Undefined
2	$\frac{1}{2-1} = 1$

B. Find a particular solution $y = f(x)$ to the differential equation with the initial condition $(-1, 0)$ and state its domain.

Since $\frac{dy}{dx} = \frac{1}{x-1}$ is undefined at $x = 1$, the domain of $y = f(x)$ must be either $x < 1$ or $x > 1$. Given the initial condition $(-1, 0)$, the domain will be restricted to $x < 1$.

$$\frac{dy}{dx} = \frac{1}{x-1},\ dy = \frac{1}{x-1}dx,\ \int dy = \int \frac{1}{x-1}dx, \text{ and }\ y = \ln|x-1| + C$$

Now given $(-1, 0)$, $0 = \ln|-2| + C$, $0 = \ln(2) + C$, $C = -\ln(2)$

So, $y = \ln|x-1| - \ln(2)$ or $y = \ln\left|\frac{x-1}{2}\right|, x < 1$

Note: This is also equivalent to $y = \ln\left(\frac{-x+1}{2}\right)$ and still has a domain of $x < 1$.

Tie Breaker 2 Answer

On Tie Breaker #1, show work and provide exact values of solutions.

For $0 \le t \le 6$ seconds, a screen saver on a computer screen shows two circles that start as dots and expand outward.

A. At the instant that the first circle has a radius of 9 centimeters, the radius is increasing at a rate of $\frac{3}{2}$ centimeters per second. Find the rate at which the area of the circle is changing at that instant. Indicate units of measure.

$$A = \pi r^2, \frac{dA}{dt} = 2\pi r\frac{dr}{dt}, \text{ evaluated with } r = 9\ cm \text{ and } \frac{dr}{dt} = \frac{3}{2}\frac{cm}{sec},$$

$$\frac{dA}{dt} = 2\pi(9\ cm)\left(\frac{3}{2}\frac{cm}{sec}\right) = 27\pi\frac{cm^2}{sec}.$$

When the radius is 9 cm, the area is changing at a rate of $27\pi\frac{cm^2}{sec}$.

B. The radius of the first circle is modeled by $w(t) = 12 - 12e^{-0.5t}$ for $0 \le t \le 6$, where $w(t)$ is measured in centimeters and t is measured in seconds. At what time t is the radius of the circle increasing at a rate of 3 centimeters per second?

$$w(t) = 12 - 12e^{-0.5t},\ w'(t) = (-0.5)(-12)e^{-0.5t} = 6e^{-0.5t}$$

$$6e^{-0.5t} = 3, e^{-0.5t} = \frac{1}{2},\quad -0.5\,t = \ln\left(\frac{1}{2}\right) \text{ or } -\ln(2),\quad -0.5\,t = -\ln(2),\qquad t = 2\ln 2$$

The radius is increasing at a rate of $3\ \frac{cm}{sec}$ at the time $t = 2\ln 2$ seconds.

C. A model for the radius of the second circle is given by the function f for $0 \le t \le 6$, where $f(t)$ is measured in centimeters and t is measured in seconds. The rate of change of the radius of the second circle is given by $f'(t) = t^2 - 4t + 4$. Based on this model, by how many centimeters does the radius of the second circle increase from time $t = 0$ to $t = 3$?

$$\int_0^3 (t^2 - 4t + 4)\,dt = \left[\frac{1}{3}t^3 - 2t^2 + 4t\right]_{t=0}^{t=3}$$

$$= \left(\frac{1}{3}(3)^3 - 2(3)^2 + 4(3)\right) - \left(\frac{1}{3}(0)^3 - 2(0)^2 + 4(0)\right) = 9 - 18 + 12 = 3$$

The radius increases by 3 cm from time $t = 0$ to $t = 3$ seconds.

Tie Breaker 3 Answer

Consider the function $f(x) = \ln(x) + 1$

a) What point on the graph of $f(x) = \ln(x) + 1$ is closest to the point $(3, 0)$. Round your coordinates to the nearest thousandth.

The distance, d, from a point $(x, \ln(x) + 1)$ and the point $(3, 0)$ is $d = \sqrt{(\ln(x) + 1)^2 + (3 - x)^2}$ and d will be a minimum when $g(x) = (\ln(x) + 1)^2 + (3 - x)^2$ is a minimum.

(Similarly, d will be a minimum when $d^2 = (\ln(x) + 1)^2 + (3 - x)^2$ is a minimum.)

$$g'(x) = \frac{2(\ln(x) + 1)}{x} - 2(3 - x)$$

Solve $\frac{2(\ln(x)+1)}{x} - 2(3 - x) = 0$ graphically or with a solve feature of a graphing calculator yields $x \approx 2.184539$ and then $y = f(x) = \ln(2.184539) + 1 = 1.78141$.

So, to the nearest thousandth, the point $(2.185, 1.781)$ is the closest point on the graph of $f(x) = \ln(x) + 1$ to the point $(3, 0)$.

b) Find the volume created if the area bounded by $f(x) = \ln(x) + 1$, $x = 3$, and $y = 0$ is rotated around the y-axis. Write the integral used to find your answers. Round your answer to the nearest thousandth.

Washer method: Solving $y = \ln(x) + 1$ for x yields $x = e^{y-1}$.

$$\pi \int_0^{\ln(3)+1} (3^2 - (e^{y-1})^2)\, dy \approx 45.412$$

or

Shell Method

$$2\pi \int_{e^{-1}}^{3} (x)(\ln(x) + 1)\, dx \approx 45.412$$