Arkansas Council of Teachers of Mathematics

Work on the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then, attempt the tiebreaker questions at the end starting with Tie Breaker #1, then #2, and finally #3. Turn in your answer sheet and the tiebreaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Figures aren't necessarily drawn to scale. Angles are given in radians unless otherwise stated. Assume all values are real.

1. Evaluate $\lim_{x\to\frac{\pi}{3}}\left(\frac{\cos(2x)-\frac{1}{2}}{x-\frac{\pi}{3}}\right)=$

   A. 0

   B. $\frac{\sqrt{3}}{2}$

   C. $\sqrt{3}$

   D. $-\sqrt{3}$

   E. Does not exist

2. Consider the differentiable function f on the closed interval $[1,4]$ with $f(1)=15$ and $f(4)=5$. Which of the following statements must be true?

   A. $f'(x)$ is negative on $(1,4)$.

   B. $5 \le f(3) \le 15$

   C. $f'(x)=-\frac{10}{3}$ has at least one solution in the interval $(1,4)$.

   D. $f(x)=3$ has at least one solution in the interval $[1,4]$.

   E. All of the above

3. Let $f(x)$ and $g(x)$ be continuous on $(-\infty,\infty)$. Which of the following indicates that the graph of $y=\frac{f(x)}{g(x)}$ has a removable discontinuity at $x=5$?

   A. $\lim_{x\to5^-}\frac{f(x)}{g(x)}=\lim_{x\to5^+}\frac{f(x)}{g(x)}$

   B. $g(5)=0$ & $\lim_{x\to5}\frac{f(x)}{g(x)}=7$

   C. $g(5)=0$ & $f(5)=0$

   D. Both A and B

   E. A, B, and C

4. Find the horizontal asymptote(s) on $f(x) = \frac{\sqrt{25x^2+7} + 110}{3x + 40}$.

   A. $y = 2$

   B. $y = \frac{5}{3}$

   C. $y = 110$

   D. $y = \frac{5}{3}$ and $y = -\frac{5}{3}$

   E. No horizontal asymptotes; $f(x)$ is always decreasing.

5. What is the average value of $f(x) = 2x^3 + x$ on the interval $[-2, 3]$?

   A. 7

   B. 9.75

   C. 15

   D. 35

   E. None of the above

6. Which of the following is an equation of the line normal to the curve $f(x) = x^2 + 3x$ when $x = 1$?

   A. $y = 5x - 1$

   B. $y = -5x + 9$

   C. $y - 4 = \frac{1}{5}(x - 1)$

   D. $y = -\frac{1}{5}x + \frac{21}{5}$

   E. None of the above

7. If the position of an object at time t is $s(t) = t^3 - t^2 - 5t + 11$, when is the acceleration increasing?

   A. $t \leq -1,\ t \geq \frac{5}{3}$

   B. $-1 \leq t \leq \frac{5}{3}$

   C. $t \geq \frac{1}{3}$

   D. $t \leq \frac{1}{3}$

   E. For all values of t.

8. Evaluate $\lim_{x \to \frac{\pi}{6}} \left(\frac{\sec(5x) + \frac{2}{\sqrt{3}}}{x - \frac{\pi}{6}} \right) =$

   A. $\frac{10}{3}$

   B. $-\frac{2}{\sqrt{3}}$

   C. $\frac{20\sqrt{3}}{3}$

   D. $\frac{20}{3}$

   E. Does not exist

9. The graph of $f(x)$ is shown to the right.

   Which of the following limits exist?

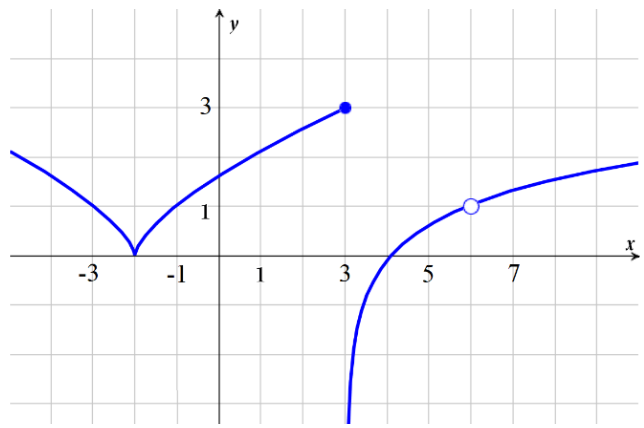
   A. $\lim_{x \to -2} \frac{f(x) - f(-2)}{x + 2}$

   B. $\lim_{x \to 3} f(x)$

   C. $\lim_{x \to 6} \frac{f(x) - 1}{x - 6}$

   D. A & C

   E. None of the Above



10. The *hyperbolic cosine* function is defined as $cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$. Evaluate $\frac{d}{dx}[cosh(x^2)]$.

   A. $\frac{1}{2}(e^{x} + e^{-x})$

   B. $\frac{1}{2}(e^{x} - e^{-x})$

   C. $x\left(e^{x^2} + e^{-x^2}\right)$

   D. $x\left(e^{x^2} - e^{-x^2}\right)$

   E. Can not be determined

11. Consider the function $f(x) = 3x^{2/3} - 2x$ on the interval $[-1, 8]$. Which of the following is true?
    - A. Mean Value Theorem applies, but the Mean Value Theorem for Integrals does not apply.
    - B. Mean Value Theorem for Integrals applies, but the Mean Value Theorem does not apply.
    - C. Both the Mean Value Theorem and the Mean Value Theorem for Integrals apply.
    - D. Neither the Mean Value Theorem nor the Mean Value Theorem for Integrals apply.
    - E. Can not be determined with the given information.

12. The derivative of the function f is continuous on the closed interval $[1, 6]$. Values of f and f' for selected values of x are given in the table below. Calculate $\int_1^6 f'(t)\,dt$.
    - A. -2
    - B. -5
    - C. 5
    - D. 2
    - E. Not enough information given

x	1	2	4	6
$f(x)$	5	2	5	3
$f'(x)$	-3	1	3	4

13. The position, in meters, of an object is given by $s(t) = t^4 - 6t^2 + 2t$ at t seconds. What is the average velocity from $t = 1$ to $t = 4$?
    - A. $108\ \frac{m}{s}$
    - B. $102\ \frac{m}{s}$
    - C. $82.5\ \frac{m}{s}$
    - D. $72\ \frac{m}{s}$
    - E. $57\ \frac{m}{s}$

14. Write an equation of the line tangent to the graph of $\sin(y) + \cos(x) = 1$ at the point $\left(\frac{\pi}{3}, \frac{\pi}{6}\right)$.
    - A. $y = \frac{\pi}{6}$
    - B. $y = \frac{\sqrt{3}}{2}$
    - C. $y = x - \frac{\pi}{6}$
    - D. $y - \frac{\pi}{6} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$
    - E. None of the above

15. Values on the continuous differentiable functions $f(x)$ and $f'(x)$ are given below.

If $g(x) = f^{-1}(x)$ what is the value of $g'(5)$?

x	0	1	2	5
$f(x)$	7	2	5	3
$f'(x)$	-3	5	3	4

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. 4

D. $\frac{1}{4}$

E. Not enough information given

16. Evaluate $\lim\limits_{\Delta x \to 0} \frac{e^{(x+\Delta x)^3} - e^{x^3}}{\Delta x} =$

A. 0

B. $3x^2 e^{x^3}$

C. $ln\left(e^{x^3}\right)$

D. $3x^2 ln(x^3)$

E. Does not exist

17. Find the following indefinite integral: $\int \left(x \tan(x^2)\right) dx =$

A. $-\frac{1}{2}\sec^2(x^2) + C$

B. $-\frac{1}{2}\sec(x^2)\tan(x^2) + C$

C. $-\frac{1}{2} ln|sin(x^2)| + C$

D. $-\frac{1}{2} ln|cos(x^2)| + C$

E. None of the above

18. Solve the differential equation $\frac{dy}{dt} = 3y - 6$ such that $y(0) = 5$.

A. $y = ln|3y - 6| + 7$

B. $y = \frac{1}{3} ln|y - 2| + 7$

C. $y = 3e^{3t} + 2$

D. $y = \frac{1}{3} e^{3t} + 7$

E. None of the above.

19. Calculate the total area bounded by the curves $f(x) = \sin(x)$ and $g(x) = \cos(x)$ between $x = \frac{\pi}{4}$ and $x = \frac{9\pi}{4}$.

    A. 0
    B. $2\sqrt{2}$
    C. $2\pi\sqrt{2}$
    D. $4\sqrt{2}$
    E. None of the Above

20. If the area bounded by $f(x) = x^2 + 1$ and $g(x) = 1 + 6x - x^2$ is rotated around the horizontal line $y = -1$, what is the volume of the three-dimensional object created?

    A. 117π
    B. 81π
    C. $\frac{102}{5}\pi$
    D. $\frac{162}{5}\pi$
    E. None of the Above

21. Use the table below to approximate $\int_{-2}^{14} g(x)\,dx$. Use a midpoint approximation with 4 equal subintervals.

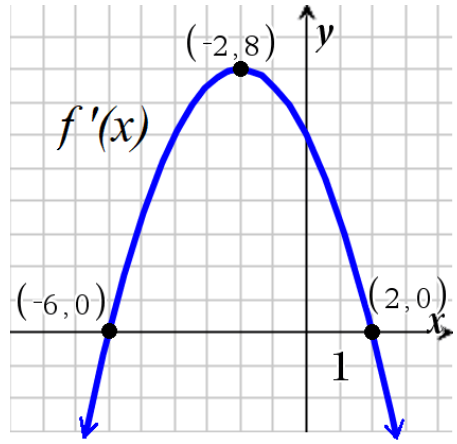
x	-2	0	2	4	6	8	10	12	14
$f(x)$	8	13	7	6	9	-3	-7	-10	-2

    A. 24
    B. 36
    C. 48
    D. 68
    E. None of the above

22. The velocity of a particle moving along a line is given by $v(t) = 5 - t^2$. If the position at time $t = 4$ is 15, what was the position at $t = 1$.

    A. $s(1) = 6$
    B. $s(1) = -6$
    C. $s(1) = 9$
    D. $s(1) = 21$
    E. None of the above

23. Which of the following must be true based on the given graph of $f'(x)$ graphed to the right?

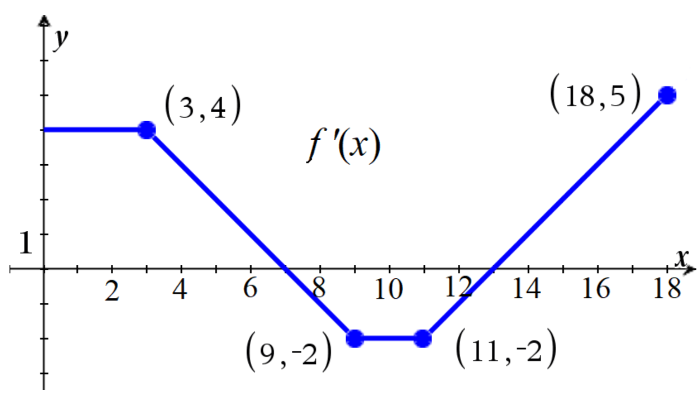


A. $f(x)$ is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$
B. $f(x)$ has an inflection point at $x = -2$
C. $f(x)$ has inflection points at $x = -6$ and $x = 2$
D. $f(x)$ has a relative min at $x = -6$ and a relative max $x = 2$
E. Both B and D

24. Given $f(x) = \frac{2x^3-6x^2}{x-3}$, the $\lim_{x\to 3} f(x) = 18$. By the definition of a limit there exists a positive real number δ such that $|f(x) - 18| < 0.25$ for each x where $0 < |x - 3| < \delta$. The largest valid value of δ, rounded to the nearest hundredth, is

A. 0.05
B. 0.04
C. 0.03
D. 0.02
E. 0.01

25. The graph of $y = f'(x)$ is graphed to the right. If $f(4) = 5$, what is the value of $f(18)$?



A. -9
B. 9
C. 14
D. 25
E. 30

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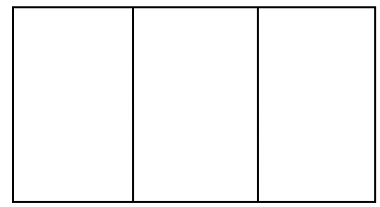
Tie Breaker #1

Name: ___________________

School: __________________

A rancher has 300 feet of fencing with which to build 3 equal adjacent rectangular corrals. What should the dimension of each corral be to maximize the enclosed area? Show all your work.



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Tie Breaker #2

Name: __________________

School: _________________

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is $55°F$. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

b) Use the data in the table to evaluate $\int_0^{20} W'(t)\, dt$.

Using correct units, interpret the meaning of $\int_0^{20} W'(t)\, dt$ in the context of this problem.

c) For $20 \leq t \leq 25$, the function W that models the water temperature has first derivative given by $W(\text{t}) = 0.4\sqrt{t} \cdot \cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

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Tie Breaker #3

Name: __________________

School: _________________

The Maclaurin series (Taylor series centered at $x = 0$) for $f(x) = \sin(x)$ is given below. Use the first 4 terms and general term to answer the given questions.

Note: You can apply your knowledge to answer the questions even if you have not studied Taylor series.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

a) Write the first 4 terms and the general term of the Taylor polynomial for $g(x) = \cos(x)$.

b) Write the first 4 terms and the general term of the Taylor polynomial for $h(x) = \cos(x^2)$.

Name ______________________________

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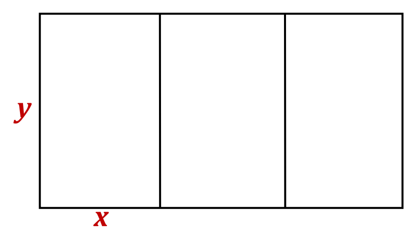
Multiple Choice Solutions to 2024 Regional Calculus Exam

1	E		11	B		21	A
2	C		12	A		22	D
3	B		13	E		23	E
4	D		14	C		24	D
5	A		15	B		25	C
6	D		16	B			
7	E		17	D			
8	A		18	C			
9	C		19	D			
10	D		20	A			

Tie Breaker Solutions to 2024 Regional Calculus Exam

Tie Breaker 1 Answer

A rancher has 300 feet of fencing with which to build 3 equal adjacent rectangular corrals. What should the dimension of each corral be to maximize the enclosed area? Show all your work.



$6x + 4y = 300$ So, $\frac{3}{2}x + y = 75$, and $y = 75 - \frac{3}{2}x$

The enclosed area will be a max when the area of each corral is a max. $A = xy$

Using the fencing restrictions $y = 75 - \frac{3}{2}x$, $A = x\left(75 - \frac{3}{2}x\right) = 75x - \frac{3}{2}x^2$

$$\frac{dA}{dx} = 75 - 3x,\ 75 - 3x = 0, \text{ and } x = 25$$

Substituting to find y,

$$y = 75 - \frac{3}{2}(25) = \frac{75}{2} \ or\ 37.5$$

The dimensions of each corral should be $25ft \times 37.5ft$ to maximize the area.

Tie Breaker 2 Answer

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is $55°F$. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$.

Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

c) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W(\mathrm{t}) = 0.4\sqrt{t} \cdot \cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

a) $W'(12) \approx \frac{W(15)-W(9)}{15-9} = \frac{67.9-61.8}{6} = \frac{6.1}{6} \approx 1.017$

So, at $t = 12$ minutes, the water temperature is increasing at a rate of approximately 1.017°F per minute.

b) $\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16$. **So, the water warmed up 16°F from $t = 0$ to $t = 20$ minutes.**

c) $W(25) = 71.0 + \int_{20}^{25} W'(t)\,dt = 71.0 + 2.043155 = 73.043$

Tie Breaker 3 Answer

The Maclaurin series (Talyor series centered at $x = 0$) for $f(x) = \sin(x)$ is given below. Use the first 4 terms and general term to answer the given questions. Note: You can apply your knowledge and answer the questions even if you have not studied Taylor series.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

a) Write first 4 terms and general term of the Taylor polynomial for $g(x) = \cos(x)$.

b) Write first 4 terms and general term of the Taylor polynomial for $h(x) = \cos(x^2)$.

a)

$$g(x) = \cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= 1 - \frac{3x^3}{3 \cdot 2!} + \frac{5x^4}{5 \cdot 4!} - \frac{7x^6}{7 \cdot 6!} + \cdots + \frac{(2n+1)(-1)^n x^{2n}}{(2n+1) \cdot 2n!} + \cdots$$

$$= 1 - \frac{x^3}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{2n!} + \cdots$$

b) **Evaluate. If $f(x) = \cos(x)$, $h(x) = f(x^2) = \cos(x^2)$ and**

$$\cos(x^2) = 1 - \frac{(x^2)^3}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots + \frac{(-1)^n (x^2)^{2n}}{2n!} + \cdots$$

$$= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + \frac{(-1)^n x^{4n}}{2n!} + \cdots$$