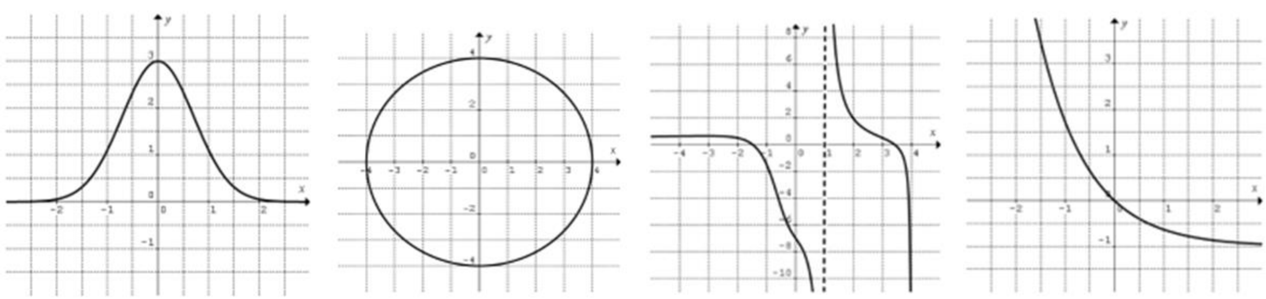
<u>2024 Regional Algebra II Competition</u> – Page 1 **Name** ______________________________
Arkansas Council of Teachers of Mathematics

Work on the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then attempt the tie-breaker questions at the end starting with tie breaker #1, then #2, and then #3. Turn in your answer sheet, your tie-breaker pages, and your scratch work when you are finished. Figures are not necessarily drawn to scale.

1. Consider complex numbers $z_1 = 2 + 3i$ and $z_2 = 1 + 2i$. Which of the following is true?
   a. $z_1 - z_2 = 1 + i$
   b. $z_1 \cdot z_2 = -4 + 7i$
   c. $z_1 \cdot z_2 = 2 + 6i$
   d. A & B are both true.
   e. A & C are both true

2. The quadratic equation $3x^2 + 2x + 1 = 0$ has
   a. Two distinct real solutions
   b. Two equal real solutions
   c. Two equal complex solutions
   d. Two distinct complex solutions
   e. No solutions of any type

3. Which of the following graph(s) represent one-to-one function(s)?



(1) (2) (3) (4)

   a. 1 and 3
   b. 3 and 4
   c. 4 only
   d. 2 only
   e. All four are one-to-one functions

4. Given $f(x) = \begin{cases} x^2 - 5 \text{ if } x \leq 2 \\ 2x + 3 \text{ if } x > 2 \end{cases}$, then $f(-3)$ is equal to
   a. 4
   b. -3
   c. -3 and 4
   d. -5 and 3
   e. $f(x)$ is not a function

5. Consider $f(x) = x^2 - 1$ and $g(x) = \sqrt{x-3}$. Let $D_{f \circ g}$ represent the domain of the composition function $f \circ g$. What is $D_{f \circ g}$?
   a. $D_{f \circ g} = (-\infty, \infty)$
   b. $D_{f \circ g} = [3, \infty)$
   c. $D_{f \circ g} = (-\infty, 3) \cup (3, \infty)$
   d. $D_{f \circ g} = [-3, 3]$
   e. $D_{f \circ g}$ does not exist

6. Let $f(x) = -x^2 + 2, x \geq 0$. Then the inverse function $f^{-1}(x)$ is given by
   a. $f^{-1}(x) = \frac{1}{-x^2+2}, x \leq 2$
   b. $f^{-1}(x) = x^2 - 2, x \leq 0$
   c. $f^{-1}(x) = \sqrt{2-x}, x \leq 2$
   d. $f^{-1}(x) = -\sqrt{2-x}, x \geq 2$
   e. The f is not a one-to-one function

7. In a laboratory experiment, the data shows that the number of bacteria in a laboratory dish grows at an exponential rate of 5.4 % per week. Approximately how many months will it take for an initial population of 460 of these bacteria to quadruple itself in size? (Round to the nearest number of months.)
   a. 6 months
   b. 5 months
   c. 16 months
   d. 23 months
   e. None of the above

8. The daily cost y, in dollars, of a small toy company to produce x baby dolls is given by the quadratic function

$$y = 2000 - 15x + 0.05x^2 \text{ with } 0 \le x \le 300 \ \& \ 0 \le y \le 2500$$

How many dolls should be produced each day to keep the costs to a minimum?

  a. 125 dolls for a minimum cost of $906.25
  b. 150 dolls for a minimum cost of $875
  c. 200 dolls for a minimum cost of $1000
  d. 250 dolls for a minimum cost of $375
  e. None of the above

9. Let $\log_b x = 31$, $\log_b y = 17$, and $\log_b z = 14$, then the value of $\log_b \frac{x^2\sqrt{z}}{y^2}$ is

  a. 62
  b. 12
  c. 35
  d. 7
  e. Impossible to determine unless the value of b and x are given

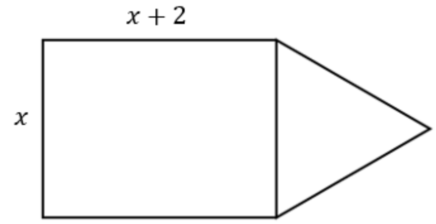
10. Which of the following is a polynomial of degree 4 with zeros at -2 (with multiplicity 2), 1, and 3?

  a. $f(x) = 2(x-2)^2(x+1)(x+3)$
  b. $f(x) = 2(x+2)^2(x-1)(x-3)$
  c. $f(x) = \frac{1}{2}(x+2)^2(x-1)(x+3)$
  d. $f(x) = -2(x+1)(x+3)(x-2)$
  e. All the above.

11. If $g(x)$ is an odd function and the point $(3, -7)$ is on its graph, then which of the following points must also be on the graph?

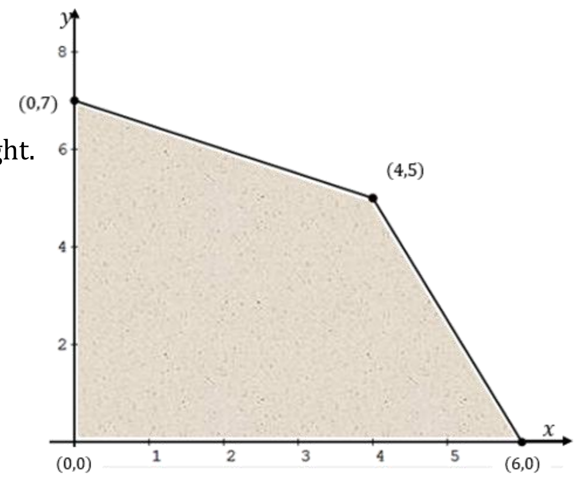
  a. $(-3, 7)$
  b. $(-3, -7)$
  c. $(-7, -3)$
  d. $(3, 7)$
  e. $(3, -7)$

12. James wants to build fencing around an area of his land to house and separate his chickens and ducks. The area has a rectangular section for the chickens and an equilateral triangular section for the ducks, as shown beside. He has budget to purchase only 76 feet of fencing. What would be the area of the section for the chickens?



   a. 236 ft^2
   b. 379 ft^2
   c. 252ft^2
   d. 168ft^2
   e. None of the above

13. Find the maximum and the minimum values of the objective function $z = 2x + 8y + 7$ over the feasible region given on the right.

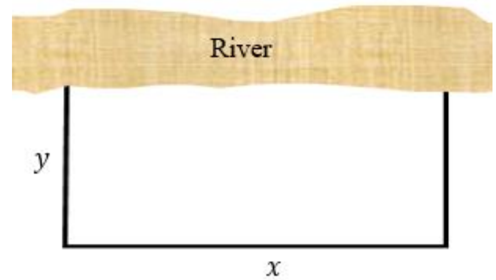


   a. Max. = 82, Min. = 7
   b. Max. = 55, Min. = 19
   c. Max. = 19, Min. = 7
   d. Max. = 63, Min. = 7
   e. No minimum or maximum values exist.

14. Consider a polynomial of degree n with real coefficients. Which of the following statements is true?
   a. The polynomial never has a complex zero.
   b. The number of complex zeros of the polynomial (if any) is an even number.
   c. The polynomial has an equal number of real and complex zeros.
   d. The polynomial has n complex zeros.
   e. The polynomial has n real zeros.

15. Consider functions $g(x) = (x + 1)^2 - 2$ and $f(x) = x^2$. The graph of g could be obtained by
   a. Shifting the graph of f 1 unit to the left and 2 units upward
   b. Shifting the graph of f 1 unit to the right and 2 units downward
   c. Shifting the graph of f 1 unit to the left and 2 units downward
   d. Rotating the graph of f clockwise by 45° and shifting 2 units downward
   e. Rotating the graph of f counterclockwise by 45° and shifting 2 units downward

16. A farmer wants to build a shed on a rectangular field and fence the field with 8000 feet of fencing. One side of the field will be a river and does not need fencing. Find the dimensions x and y of the field so that the area enclosed is maximum.



   a. $x = 5000$ ft, $y = 1500$ ft
   b. $x = 6000$ ft, $y = 1000$ ft
   c. $x = 4000$ ft, $y = 2000$ ft
   d. $x = 3000$ ft, $y = 2500$ ft
   e. Does not make an area

17. In year 2015, the population of country Rokanda was 150 million and it has been growing exponentially at an annual rate of 0.523%. In the same year, the population of country Barenda was 310 million and it has been decaying exponentially at an annual rate of 0.375%. In what year will the two countries have equal populations? What would that population be?
   a. Year 2096, Population $=$ 229 million
   b. Year 2023, Population $=$ 229 million
   c. Year 2032, Population $=$ 164 million
   d. Year 2028, Population $=$ 295 million
   e. They will be equal in some other year.

18. Find the value of b with $b > 0$ for which the function $f(x) = b^x$ passes though point $(-2, 4)$.
   a. $b = -\frac{1}{2}$
   b. $b = 2$
   c. $b = \frac{1}{2}$
   d. $b = 4$
   e. No value for b exists

19. Solve the equation: $5^{x-1} = 39$

   a. $x = 8$
   b. $x = 1.3225$
   c. $x = 1.8921$
   d. $x = 3.2763$
   e. The equation cannot be solved

20. A company manufactures two models of deer stands. The deluxe model requires 6 hours of assembly time and 3 hours of finishing time. The economy model requires 3 hours of assembly time and 2 hours of finishing time. Each week the company has at most 150 hours of assembly time and not more than 90 hours of finishing time available for production. If x is the number of deluxe and y is the number of economy stands produced, which set of inequalities represents this situation?

   a. $\begin{cases} 6x + 3y \le 150 \\ 2x + 3y \le 90 \\ x \ge 0, y \ge 0 \end{cases}$
   b. $\begin{cases} 6x + 3y \le 90 \\ 2x + 3y \le 150 \\ x \ge 0, y \ge 0 \end{cases}$
   c. $\begin{cases} 6x + 3y \le 150 \\ 3x + 2y \le 90 \\ x \ge 0, y \ge 0 \end{cases}$
   d. $\begin{cases} 2x + 3y \le 150 \\ 3x + 6y \le 90 \\ x \ge 0, y \ge 0 \end{cases}$
   e. None of the above.

21. Find the solutions of the matrix equation:

$$\begin{bmatrix} 2x + y & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 4 & x - y \end{bmatrix}$$

   a. $x = -2, y = -1$
   b. $x = 1, y = 3$
   c. $x = 2, y = -1$
   d. $x = 1, y = -2$
   e. $x = 2, y = 1$

22. Which of the following is not a property of exponential function $f(x) = a^x$, $0 \leq a \leq 1$?
   a. The domain is $(-\infty, \infty)$
   b. The range is $(0, \infty)$
   c. The y-intercept is $(0, 1)$
   d. The graph of the function passes though point $\left(-1, \frac{1}{a}\right)$
   e. The function is increasing

23. Find the vertical asymptote(s) of the function $f(x)$.

$$f(x) = \frac{x+4}{x^2 - 4x + 3}$$

   a. $x = 3$
   b. $x = 1$ and $x = 3$
   c. $x = -1$ and $x = -3$
   d. $x = -1$ and $x = 3$
   e. $x = 1$ and $x = -3$

24. For what value of a does the rational function $f(x) = \frac{ax^2-4}{2x^2-3x}$ have a horizontal asymptote of $y = 3$?
   a. $a = 3$
   b. $a = -6$
   c. $a = -3$
   d. $a = 6$
   e. $a = 2$

25. Express $2\log x + \frac{1}{3}\log y$ as a single logarithm.
   a. $\log\left(2x + \frac{1}{3}y\right)$
   b. $\log\left(x^2 + \sqrt[3]{y}\right)$
   c. $\log\left(x^2\sqrt[3]{y}\right)$
   d. $\log\left(\frac{2}{3}xy\right)$
   e. Not possible with given information

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Tie Breaker #1

Name: ____________________

School: ___________________

Let $f(x) = -1 + \log_2(x+4)$.

a. Evaluate $f(4)$.

b. Find the domain of f.

c. Does $f(x)$ have a vertical asymptote? If yes, what is it and why?

d. Find the y-intercept(s) and x-intercept(s) of f. (If an intercept does not exist, explain why.)

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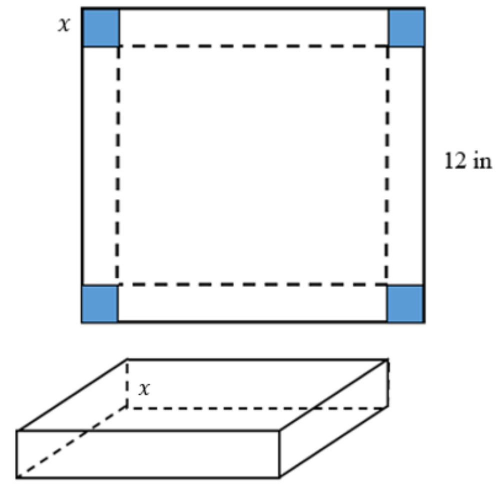
Tie Breaker #2



Name: __________________

School: _________________

An open box is to be made from a square piece of material 12 inches on each side. This is done by cutting squares from the corners and folding along the dotted line (see the figure).

a. Write expressions for the length, width, and height of the open box. Use x as your variable.

b. Use your expressions in part a to define a function for the volume of the box.

c. What is the domain of your volume function found in part b?

d. Find the dimensions of the square cut from the corners which maximizes the volume of the box.

Name ______________________________

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Tie Breaker #3

Name: __________________

School: _________________

Find the value(s) of x for which this matrix has an inverse.

$$\begin{bmatrix} 3x-4 & 3 \\ 2x-1 & x \end{bmatrix}$$

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ANSWER KEY

1) D	11) A	21) E
2) D	12) D	22) E
3) C	13) D	23) B
4) A	14) B	24) D
5) B	15) C	25) C
6) C	16) C	
7) A	17) A	
8) B	18) C	
9) C	19) D	
10) B	20) C	

TB1:

a. $f(4) = -1 + \log_2(4+4) = -1 + \log_2(8) = -1 + 3 = 2$

b. Domain: We must have $x + 4 > 0$, so $x > -4$, or Domain $(-4, \infty)$

c. Vertical Asymptote: VA at $x = -4$, because the function is undefined at $x = -4$.

d. Intercepts:

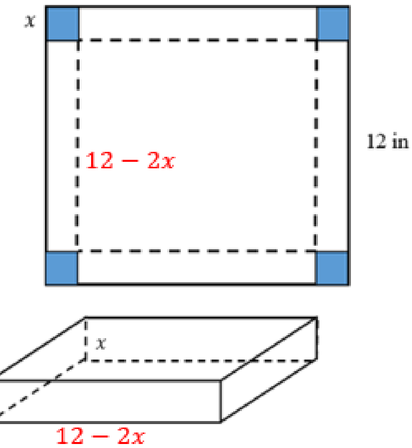
y-intercept= $f(0) = -1 + \log_2(0+4) = -1 + 2 = 1$

For the x-intercept, set $-1 + \log_2(x+4) = 0$ or $\log_2(x+4) = 1$

This implies $(x+4) = 2$. So, $x = -2 = x$-intercept.

Alternately: y-intercept at $(0, 1)$; x-intercept at $(-2, 0)$.

TB2:



a. Answers may vary, but generally,
$\boldsymbol{length} = 12 - 2x, \quad \boldsymbol{width} = 12 - 2x, \quad \boldsymbol{height} = x$

b. $V(x) = x(12 - 2x)^2$

c. Domain $D = [0, 6]$ or $0 \le x \le 6$ or similar.

d. Upon graphing this function for the given domain gives the graph on the lower right. From the graph we get $x = 2$ inches, and $V_{\max} = 128 \text{ in}^3$.

TB3:

For the matrix to have an inverse, it's determinant must not be equal to zero. We must solve $\det\left(\begin{bmatrix} 3x-4 & 3 \\ 2x-1 & x \end{bmatrix}\right) \neq 0$.

So, we set the determinant equal to zero and solve to find the "forbidden" values.

$$\begin{aligned} \det\left(\begin{bmatrix} 3x-4 & 3 \\ 2x-1 & x \end{bmatrix}\right) &= (3x-4)x - 3(2x-1) \\ &= 3x^2 - 10x + 3 \\ &= 0 \end{aligned}$$

We solve $(3x - 1)(x - 3) = 0$ to yield $x = \frac{1}{3}$ and $x = 3$.

Any other value of x would result in a matrix with an inverse.

The valid values of x are any values other than these two. This can be written in a few ways.

Here are two options.

$$\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 3\right) \cup (3, \infty) \quad \text{or} \quad \left\{x \mid x \neq \frac{1}{3} \text{ or } x \neq 3\right\}$$

The two specific matrices that do <u>not</u> have an inverse arise when we apply the "forbidden" x values to the given matrix $\begin{bmatrix} 3x-4 & 3 \\ 2x-1 & x \end{bmatrix}$:

$$\begin{bmatrix} 3 \cdot \frac{1}{3} - 4 & 3 \\ 2 \cdot \frac{1}{3} - 1 & \frac{1}{3} \end{bmatrix} \quad \& \quad \begin{bmatrix} 3 \cdot 3 - 4 & 3 \\ 2 \cdot 3 - 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \& \quad \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix}$$