Arkansas Council of Teachers of Mathematics **Name** ______________________________

2023 State Geometry Competition

Begin by removing the three tie breaker sheets at the end and writing your name on all three pages. Work the multiple-choice questions first, choosing the single ***best*** (most detailed and complete correct) response from the choices available. Indicate your answer here and on your answer sheet. Make sure you attempt the tie-breaker questions at the end of the test starting with tie breaker 1, then 2, and then 3 if you have time. Turn in your answer sheet and the tie breaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Notations and Definitions:

- All questions on this test are in **Euclidean Geometry** unless indicated otherwise.
- All angles are measured in **radians**. $\boldsymbol{\pi}$ **radians** $= \mathbf{180°}$.
- AB indicates the distance between points A and B.
- $\mathbf{A - B - C}$ indicates that B is ***between*** A and C. That is: A, B, and C are collinear and AB = BC.
- A ***cyclic polygon*** is a polygon for which there is a single *circumscribed* circle containing all the vertices of the polygon.
- A ***kite*** is a quadrilateral with at least two non-overlapping pairs of congruent consecutive sides. Its ***major diagonal*** has endpoints where the congruent sides meet.
- An ***isosceles trapezoid*** is a trapezoid with a pair of congruent base angles.
- An ***isometry*** (rigid transformation) is a transformation mapping every preimage to a congruent image.
- ***Z*** **Property**: Alternate interior angles formed by a transversal to lines *l* and *m* are congruent if and only if *l* and *m* are parallel

Test begins on the next page. Don't open this document until time begins.

1. Every isometry can be formed by the composition of at most _____ reflections.
   A. 1
   B. 2
   C. 3
   D. 4
   E. Each of the other answers is incorrect.

2. The composition of two reflections about parallel lines is a single _____.
   A. Rotation
   B. Reflection
   C. Translation
   D. Glide-Reflection
   E. Dilation

3. A quadrilateral has rotational symmetry about the midpoint of a diagonal if and only if it is a _____?
   A. Parallelogram
   B. Rectangle
   C. Rhombus
   D. Kite
   E. Isosceles Trapezoid

4. What is the composition of a translation by vector $\langle 3, 2\rangle$ followed by the inverse of a translation by the vector $\langle 4, 3\rangle$?
   A. A translation by vector $\langle 7, 5\rangle$
   B. A translation by vector $\langle -1, -1\rangle$
   C. A translation by vector $\langle 1, 1\rangle$
   D. A translation by vector$\langle -7, 5\rangle$
   E. Each of the other answers is incorrect.

5. A cyclic polygon is a polygon that can be inscribed in a circle. The center of the circumscribed circle is at the common intersection of the _____.
   A. bisectors of the interior angles
   B. perpendicular bisectors of the sides
   C. diagonals
   D. medians
   E. Each of the other answers is incorrect.

6. A triangle with at least two lines of reflective symmetry is a(n) _____.
   A. Right Triangle
   B. Isosceles Triangle
   C. Equilateral Triangle
   D. Scalene Triangle
   E. Each of the other answers is incorrect.

7. Which of the following types of quadrilaterals is always cyclic (that is, can always be inscribed in a circle)?
   A. Parallelogram
   B. Kite
   C. Rhombus
   D. Isosceles Trapezoid
   E. Each of the other answers is incorrect.

8. A quadrilateral has two pair of opposite congruent sides if and only if it is a _____.
   A. Parallelogram
   B. Kite
   C. Rhombus
   D. Rectangle
   E. Each of the other answers is incorrect.

9. A quadrilateral is a trapezoid with a pair of congruent diagonals if and only if it is a(n) _____.
   A. Parallelogram
   B. Rectangle
   C. Isosceles Trapezoid
   D. Square
   E. Each of the other answers is incorrect.

10. A quadrilateral has reflective symmetry about the line containing midpoints of at least one pair of opposite sides if and only if it is a(n) _____.
    A. Isosceles Trapezoid
    B. Rhombus
    C. Rectangle
    D. Kite
    E. Parallelogram

11. The sum of the measures of the exterior angles of a convex n-gon is _____.
    A. $(n-2)\frac{\pi}{2}$
    B. $n\pi$
    C. 2π
    D. $(n-2)\cdot\pi$
    E. Each of the other answers is incorrect.

12. A quadrilateral has symmetry about a line containing a pair of opposite vertices if and only if it is a _____.
    A. Parallelogram
    B. Kite
    C. Rhombus
    D. Rectangle
    E. Square

13. Triangles $\Delta ABC \cong \Delta ADC$ if and only if Quadrilateral $\Diamond ABCD$ is a(n) _____.
    A. Rectangle
    B. Trapezoid
    C. Parallelogram
    D. Isosceles Trapezoid
    E. Kite

14. A right rectangular prism has edges of length 3 cm, 4 cm, and 12 cm. What is the distance between opposite vertices?
    A. 5 cm
    B. 8 cm
    C. 13 cm
    D. 19 cm
    E. Each of the other answers is incorrect.

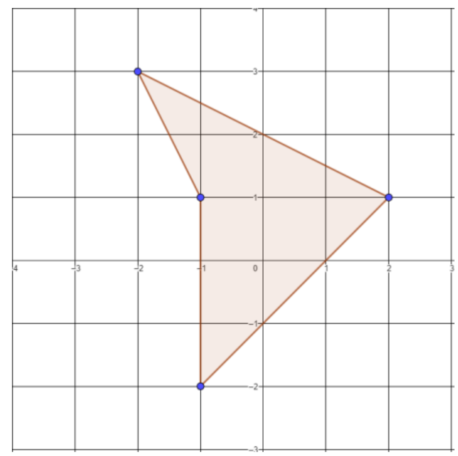
15. If we construct a diagonal of a parallelogram, then we can the two triangles formed are _____.
    A. Congruent by the Z-Property and the SAS Triangle Congruence Postulate and the Z-Property
    B. Congruent by the Z-Property and the SSS Triangle Congruence Theorem
    C. Congruent by the Z-Property and the ASA Triangle Congruence Theorem and
    D. Not necessarily congruent.
    E. Each of the other answers is incorrect.

16. Which of the following transformations has a single fixed point?
    A. Rotation
    B. Reflection
    C. Translation
    D. Glide-Reflection
    E. Identity

17. A quadrilateral has two non-overlapping pairs of consecutive congruent angles if and only if it is a(n)__________.

  A. Rhombus
  B. Kite
  C. Parallelogram
  D. Rectangle
  E. Isosceles Trapezoid

Refer to figure on the right for Questions 18 and 19:



18. What is the area of this lattice quadrilateral?

  A. 6
  B. 6.5
  C. 7
  D. 7.5
  E. Each of the other answers is incorrect.

19. What is the perimeter of this lattice quadrilateral rounded to two decimal places?

  A. 13.95
  B. 12.23
  C. 10.41
  D. 9.33
  E. Each of the other answers is incorrect.

20. Given $A - C - E$, $B - D - F$, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$, $AC = 4$, $CE = 12$, and $BD = 8$. What is DF?

    A. 4

    B. 8

    C. 12

    D. 24

    E. Each of the other answers is incorrect.

21. Given a circle X with chords $\overline{AB}$ and $\overline{CD}$ intersecting at point X, such that $AX = 3$, $BX = 8$, and $CX = 6$. What is DX?

    A. 3

    B. 4

    C. 12

    D. 16

    E. Each of the other answers is incorrect.

22. The measures of the interior angles of a triangle are x, x, and $x + x$ for some positive real number x if and only if the triangle is _____.

    A. Acute

    B. Right

    C. Isosceles

    D. Right Isosceles

    E. Each of the other answers is incorrect.

23. An equilateral triangle of side length s has an area of _____.

    A. $\frac{1}{2}s^2$

    B. $\frac{\sqrt{3}}{2}s^2$

    C. $\frac{\sqrt{3}}{4}s^2$

    D. $\sqrt{2}\,s^2$

    E. $\sqrt{3}\,s^2$

24. What is the volume of square pyramid with base edge of length 6 cm and lateral faces with slant height 5 cm?
    A. 48 cm^3
    B. 60 cm^3
    C. 120 cm^3
    D. 180 cm^3
    E. Each of the other answers is incorrect.

25. What is the surface area of square pyramid with base edge of length 6 cm and lateral faces with slant height 5 cm?
    A. 60 cm^2
    B. 96 cm^2
    C. 120 cm^2
    D. 156 cm^2
    E. Each of the other answers is incorrect.

 Name ______________________________

School ______________________________

<u>Tie Breaker 1</u>:

A relation is a set of ordered pairs. Its inverse relation is the relation found by reversing the ordered pairs. I.e., if (a, b) is an element of a relation, then (b, a) is an element of its inverse relation.

Now suppose that the first and second coordinates of the relation are real numbers so that we can graph the relation in a Cartesian coordinate plane.

Prove that the graph of an inverse relation is a reflection of the graph of the original relation over the line $y = x$.

This page left blank.

 Name ______________________________

School ______________________________

<u>Tie Breaker 2</u>:

Prove the legs of an isosceles trapezoid are congruent.
(You many NOT use the result from Tie Breaker 3. You may assume that you know properties of parallelograms.)

This page left blank.

 Name ______________________________

School ______________________________

Tie Breaker 3:

Prove the diagonals of an isosceles trapezoid are congruent.

(You may use the result of Tie Breaker 2.)

This page left blank.

ACTM State Geometry Competition

2023

Key

1.	C
2.	C
3.	A
4.	B
5.	B
6.	C
7.	D
8.	A
9.	C
10.	A
11.	C
12.	B
13.	E
14.	C
15.	C
16.	A
17.	E
18.	D
19.	A
20.	D
21.	B
22.	D
23.	C
24.	A
25.	B

Name **Key**

Tie Breaker 1:

A relation is a set of ordered pairs. Its inverse relation is the relation found by reversing the ordered pairs. I.e. if (a, b) is an element of a relation, the (b, a) is an element of its inverse relation. Now suppose that the first and second coordinates of the relation are real numbers so that we can graph the relation in a Cartesian coordinate plane. Prove that the graph of an inverse relation is a reflection of the graph of the original relation over the line $y = x$.

Proof:

We are given a relation R and its inverse relation I. Suppose A = (a, b) is any point in R then by definition A' = (b, a) is a point in I. We will show that A' = (b, a) is the image of A = (a, b) under a reflection about the line $y = x$. By definition, a reflection is a transformation where we are given a mirror line r and the line segment $\overline{AA'}$ where A' is the image of A is perpendicularly bisected by the mirror line r. In this case the mirror line r is $y = x$, A = (a, b), and A' = (b, a). So, we must show that the midpoint of $\overline{AA'}$ is on $y = x$ and that that $\overleftrightarrow{AA'}$ is perpendicular to $y = x$. To do this we use the midpoint and slope formulas.

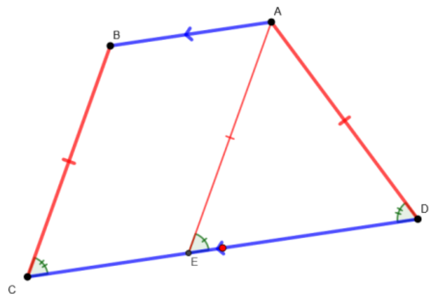
By the midpoint formula, the midpoint of $\overline{AA'}$ is $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{a+b}{2}, \frac{b+a}{2}\right) = \left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ which is on the line $y = x$.

By the slope formula, the slope of $\overleftrightarrow{AA'}$ is $m = \frac{y_2-y_1}{x_2-x_1} = \frac{b-a}{a-b} = -\frac{a-b}{a-b} = -1$. The slope of line y = x is 1. The product of these two slopes is (-1)(1) = -1, so the two lines are perpendicular. $\overline{AA'}$ is perpendicularly bisected by $y = x$. A' is an image of A under a reflection over $y = x$. Therefore, the graph of an inverse relation is a reflection of the graph of the original relation over the line $y = x$. ■

Tie Breaker 2:

Prove the legs of an isosceles trapezoid are congruent.

(You many NOT use the result from Tie Breaker 3. You may assume that you know properties of parallelograms.)



Proof:

We are given isosceles trapezoid ABCD. Without loss of generality, by the definition of an isosceles trapezoid we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and ∢C ≅ ∢D. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ the C-Property of these parallel lines cut by transversal BC tells us that ∢C and ∢B are supplementary. Similarly, ∢A and ∢D are supplementary. Thus, ∢A ≅ ∢B. So, without loss of generality we may assume that CD ≥ AB.

Case 1: If CD = AB then Quad ABCD is actually a parallelogram and thus a rectangle. We know from a previous theorem that opposite sides of a parallelogram are congruent so $\overline{AD} \cong \overline{BC}$, proving what is required.

Case 2. CD > AB. By the Parallel Postulate there exists exactly one line through A parallel to $\overleftrightarrow{BC}$. Let its intersection with $\overleftrightarrow{CD}$ be E. Construct $\overline{AE}$. By the F-Property of parallel lines $\overleftrightarrow{AE} \parallel \overleftrightarrow{BC}$ we see that corresponding angles are congruent so ∢BED ≅ ∢C. But we already had ∢C ≅ ∢D so ∢E ≅ ∢D. This makes ΔADE an isosceles triangle, and $\overline{AD} \cong \overline{AE}$. Opposite sides of parallelogram ABCE are congruent, so $\overline{AE} \cong \overline{BC}$ and thus $\overline{AD} \cong \overline{BC}$.

Therefore, in either case we have proved that the legs of an isosceles trapezoid are congruent. ■

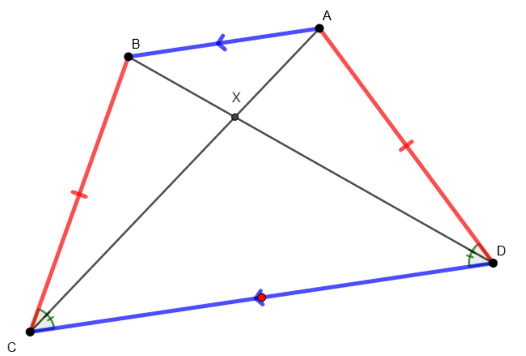
 Name ______________________________

Name ______________________________

Tie Breaker 3:

Prove the diagonals of an isosceles trapezoid are congruent.

(You may use the result of Tie Breaker 2.)



Proof:

We are given isosceles trapezoid ABCD. Without loss of generality, by the definition of an isosceles trapezoid we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and ∢C ≅ ∢D. From the results of the previous tie breaker we know that $\overline{BC} \cong \overline{AD}$. ∢C ≅ ∢D and $\overline{CD} \cong \overline{CD}$. By the SAS Triangle Congruence Postulate we know that ΔADC ≅ ΔBCD. By definition of congruent triangles corresponding parts are congruent so $\overline{AC} \cong \overline{BD}$. Therefore, the diagonals of an isosceles trapezoid are congruent. ■