**Arkansas Council of Teachers of Mathematics** **Name** ______________________

**2023 State Calculus Exam**

Work the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then, attempt the tiebreaker questions at the end starting with Tie Breaker #1, then #2, and finally #3. Turn in your answer sheet and the tiebreaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Figures aren't necessarily drawn to scale. Angles are given in radians unless otherwise stated. Assume all values are real.

1. Let $f$ be a function that is continuous on the closed interval $[1, 4]$ and differentiable on the open interval $(1, 4)$ with $f(1) = 15$ and $f(4) = 5$. Which of the following statements must be true?

   A. $5 \leq f(3) \leq 15$
   B. $f(x) = 3$ has at least one solution in the interval $[1, 4]$.
   C. $f'(x)$ is negative on $(1, 4)$.
   D. $f'(x) = -\frac{10}{3}$ has at least one solution in the interval $(1, 4)$.
   E. None of the above

2. Does the function $f(x) = \frac{\sin(x)}{x}$ have a horizontal asymptote, why?

   A. Yes, at $x = 0$ because $\lim\limits_{x \to 0^+} f(x) = \infty$
   B. No, because $\lim\limits_{x \to 0} f(x)$ does not exist.
   C. Yes, at $y = 0$ because $\lim\limits_{x \to \infty} f(x) = 0$
   D. No, because $\lim\limits_{x \to \infty} f(x)$ does not exist.
   E. No, because $f(x)$ crosses the line $y = 0$

3. Let $f(x)$ and $g(x)$ be continuous on $(-\infty, \infty)$. What would prove the graph of $y = \frac{f(x)}{g(x)}$ has point discontinuity at $x = 5$?

   A. $\lim\limits_{x \to 5} \frac{f(x)}{g(x)}$ exists.
   B. $g(5) = 0$
   C. $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$ exists.
   D. The combination of B and C
   E. The combination of A and B

4. What is the average value of $f(x) = 2x^3 - x$ on the interval $[1, 4]$ ?

   A. 40
   B. 41
   C. 62.5
   D. 120
   E. None of the above

5. $\lim_{h\to 0}\left(\frac{\cos((x+h)^2)-\cos(x^2)}{h}\right) =$

   A. $0$
   B. $\cos(2x)$
   C. $\cos(x^2)-\frac{\pi}{2}$
   D. $-2x\cdot\sin(x^2)$
   E. The limit does not exist

6. Let $G(x)$ be the antiderivative of $y=\frac{\cos(x)+x^3}{4+x^2}$. If $G(6)=5$, then $G(1)=$

   A. $-13.238$
   B. $-8.238$
   C. $13.238$
   D. $18.238$
   E. Not enough information; no value can be determined

7. The function $f$ has first derivative given by $f'(x)=x^4-6x^2+8x-10$. On what interval(s) is the graph of $f(x)$ concave down?

   A. $(-\infty,0)$
   B. $(-\infty,-2)$
   C. $(-\infty,-1)\cup(1,\infty)$
   D. $(-2,-1)\cup(1,\infty)$
   E. None of the above

8. The table at the right gives values of a function $f$ at selected values of $x$. If $f$ is twice-differentiable on the interval [2, 6], which of the following statements could be true.

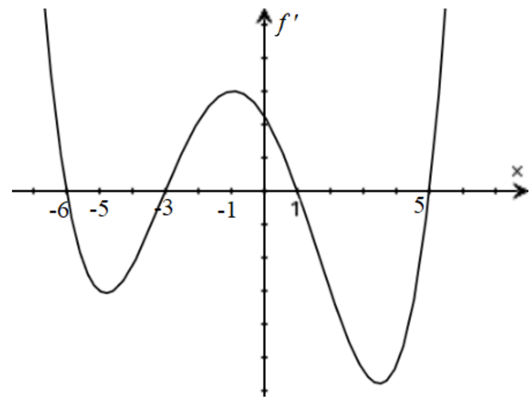
| $x$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $f(x)$ | $-7$ | $-2$ | $2$ | $5$ | $7$ |

   A. $f'$ is negative and decreasing for $2\le x\le 6$
   B. $f'$ is negative and increasing for $2\le x\le 6$
   C. $f'$ is positive and decreasing for $2\le x\le 6$
   D. $f'$ is positive and increasing for $2\le x\le 6$
   E. None of the above

9. Let $f$ be the function defined by $f(x) = \ln(x^2 + 1)$, and let $g$ be the function defined by $g(x) = x^5 + x^3$. The line tangent to the graph of $f$ at $x = 2$ is parallel to the line tangent to the graph of $g$ at $x = a$, where $a$ is a positive constant. What is the value of $a$ ?

   A. 0.246
   B. 0.430
   C. 0.447
   D. 0.790
   E. None of the above

10. The graph of the derivative of function $f$ is shown in the figure to the right. The graph has zeros at $x = -6, -3, 1,$ and $5$. The graph has horizontal tangent lines at $x = -5, -1,$ and $3.5$. At which of the following values of $x$ does $f$ have a relative maximum?




   A. $x = -6$ and $x = 1$
   B. $x = -5$ and $x = 3.5$
   C. $x = -3$ and $x = 5$
   D. $x = -1$
   E. $x = 0$

11. Given $f(x) = x^2 - 2$, find the value(s) of $c$ guaranteed by the Mean Value Theorem for integrals on the interval [-1, 5].

   A. $c = 2$
   B. $c = \sqrt{7}$
   C. $c = 5$
   D. $c = \sqrt{13}$
   E. None: MVT for integrals does not apply on this interval.

12. The velocity of an object on the $x$-axis is given by $v(t) = t^3 - 4t + 3$ at $t$ seconds. What is the position of the object at $t = 3$ if the position at $t = 1$ is 5 ?

   A. $-5$
   B. 10
   C. 15
   D. 18
   E. None of the above

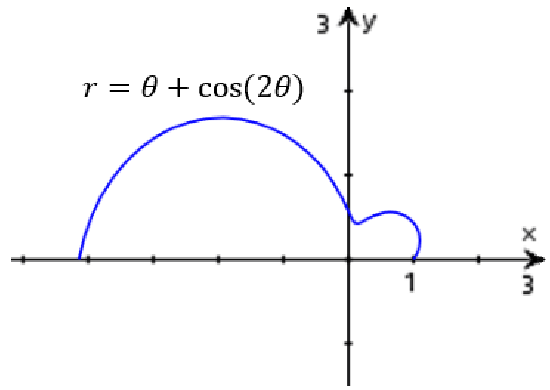
13. Solve the following differential equation $\frac{dy}{dx} = 4x - 2xy$ .

   A. $y = Ce^{-x^2} + 2$
   B. $y = Ce^{-2x} + 2$
   C. $y = Ce^{-x^2} - 2$
   D. $y = Ce^{-2x} - 2$
   E. None of the above.

14. Determine the slope of $y^3 + 2x^2y = -12$ at the point $(1, 2)$.

A. $-8$
B. $-\frac{1}{4}$
C. $-\frac{3}{4}$
D. $-\frac{4}{7}$
E. None of the above

15. The curve at the right in the $xy$-plane is described by the equation in polar coordinates $r = \theta + \cos(2\theta)$ for $0 \le \theta \le \pi$ where $\theta$ is measured in radians. What is the area bounded by the curve and the $x$-axis?




A. 2.467
B. 4.935
C. 5.953
D. 11.906
E. None of the above

16. Find the following indefinite integral: $\int (\tan x) dx =$

A. $-\ln|\cos x| + C$
B. $\sec^2 x + C$
C. $-\sec^2 x + C$
D. $\sec x \tan x + C$
E. None of the above

17. If $f(x) = \cos\left(x^2 + \frac{\pi}{2}\right)$, then $f'(\sqrt{\pi})$

A. 1
B. $-1$
C. 2
D. $2\sqrt{\pi}$
E. $\sin\left(\frac{3\pi}{2}\right)$

18. $\lim_{x \to 0} \left(\frac{5x^2}{1 - \cos(9x)}\right) =$

A. $\frac{5}{9}$
B. $-\frac{5}{9}$
C. $\frac{10}{9}$
D. $\frac{10}{81}$
E. The limit does not exist

19. Find $\int \frac{x}{\sqrt{x^2-9}}\,dx$.

 A. $\sqrt{x^2-9}+C$
 B. $2\sqrt{x^2-9}+C$
 C. $\frac{1}{2}\sqrt{x^2-9}+C$
 D. $\frac{1}{3}\arcsin(x)+C$
 E. None of the above.

20. Consider the differential equation $\frac{dy}{dx}=x^2+2y+3$. Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y.*

 A. $\frac{d^2y}{dx^2}=2x+2$
 B. $\frac{d^2y}{dx^2}=2x+2y$
 C. $\frac{d^2y}{dx^2}=2x+4y+6$
 D. $\frac{d^2y}{dx^2}=2x^2+2x+4y+6$
 E. None of the above

21. Use the table below to approximate $\int_{-2}^{14} g(x)\,dx$. Use a midpoint approximation with 4 equal subintervals.

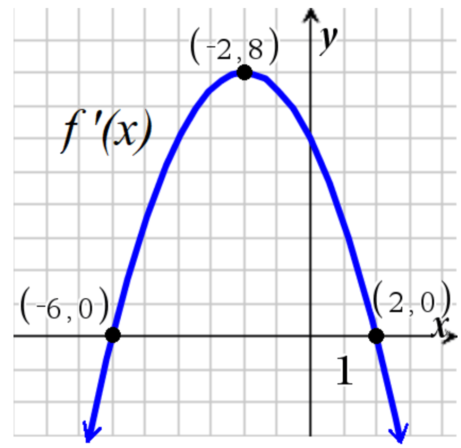
| $x$ | $-2$ | $0$ | $2$ | $4$ | $6$ | $8$ | $10$ | $12$ | $14$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $8$ | $13$ | $7$ | $6$ | $9$ | $-3$ | $-7$ | $-10$ | $-2$ |

 A. 24
 B. 36
 C. 48
 D. 68
 E. None of the above

22. $\int_2^\infty \frac{4}{x^2}\,dx =$

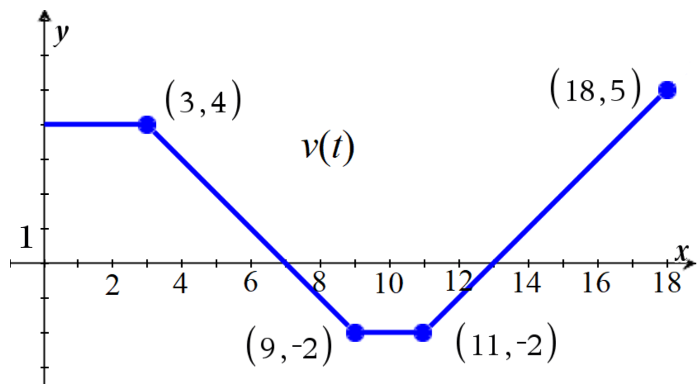
 A. 0
 B. 2
 C. $\pi$
 D. 4
 E. $\infty$

23. Which of the following must be true based on the given graph of $f'(x)$ graphed to the right?




A. $f(x)$ is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$
B. $f(x)$ has an inflection point at $x = -2$
C. $f''(x)$ changes from negative to positive at $x = -2$
D. $f(x)$ has a relative max at $x = -6$ and a relative min $x = 2$
E. None of the above

24. The graph of velocity, $v(t)$, is graphed to the right and measured in meters/sec. If the position at $t = 0$ is 10, what is the total distance travelled from $t = 0$ to $t = 18$ ?




A. 23.385
B. 24.5
C. 40.5
D. 34.5
E. 50.5

25. Find $\int x \sin x \, dx$.

A. $-\frac{1}{2}x^2 \cos x + C$
B. $\frac{1}{2}x^2 \cos x + C$
C. $x \sin x + \cos x + C$
D. $\sin x \ - x \cos x + C$
E. None of the above.

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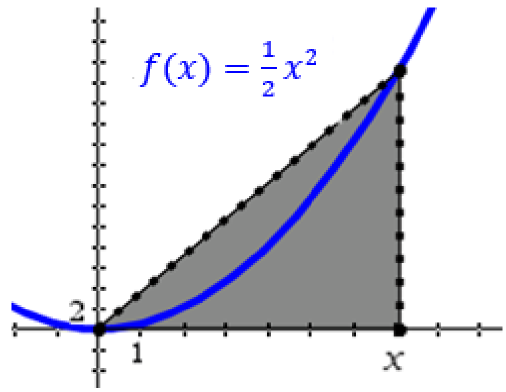
## Tie Breaker #1

A point is moving along the function $f(x) = \frac{1}{2}x^2$ such that $\frac{dx}{dt} = 3$ inches per second.

a) Find $\frac{dy}{dt}$ when $x = 5$




b) A triangle is formed when connecting the moving point to the origin and vertically to the $x$ axis as shown above. Find the rate of change of the area of the triangle when $x = 5$.

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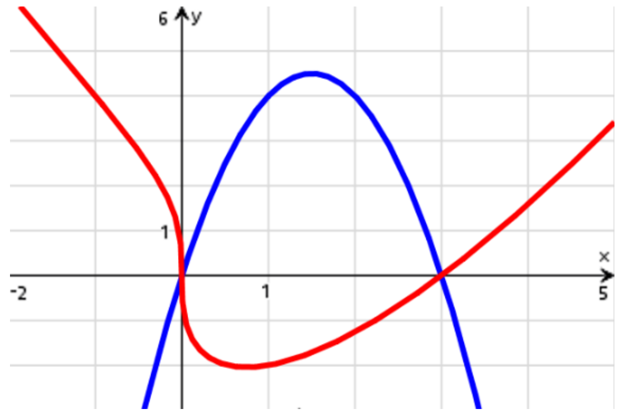
**Tie Breaker #2**

Use the functions $f(x) = 2x(3-x)$ and $g(x) = \sqrt[3]{x}(x-3)$ to answer the following problems.
Write the integral used to find your answers. Round your answers to the nearest thousandths.




a) Find the area bounded by the curves.

b) Find the volume of the solid when rotated around the line $y = 6$.

c) Find the volume of the solid when rotated around the line $x = -1$

d) Find the volume of the solid having the shaded area as its base and having rectangular cross sections with the height twice the base when taken perpendicular to the $x$-axis.

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## Tie Breaker #3

Consider the function $f(x) = x\ln(x^2+1)$.

a) Write the first four (4) non-zero terms and general term of the Maclaurin series for $g(x) = \ln(x+1)$.

b) Write the first four (4) non-zero terms and general term of the Maclaurin series for $h(x) = \ln(x^2+1)$.

c) Find the interval of convergence of the Maclaurin series for $h(x) = \ln(x^2+1)$ .

d) Write the first four (4) non-zero terms and general term of the Maclaurin series for $f(x) = x\ln(x^2+1)$.

e) Estimate $f'\left(\frac{1}{2}\right)$ using the 4 terms found on part (d). What is the error between his approximation and $f'\left(\frac{1}{2}\right)$?

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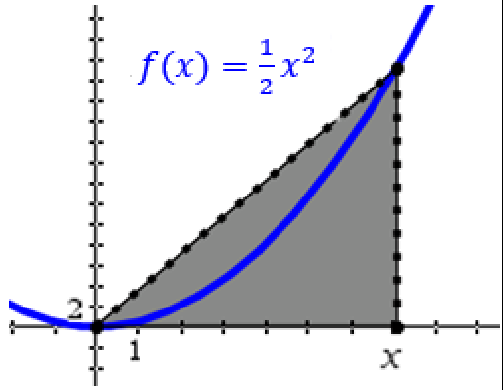
## Solutions to 2023 State Calculus Exam

| | |
|---|---|
| 1 | D |
| 2 | C |
| 3 | E |
| 4 | A |
| 5 | D |
| 6 | B |
| 7 | B |
| 8 | C |
| 9 | C |
| 10 | A |
| 11 | B |
| 12 | C |
| 13 | A |
| 14 | D |
| 15 | C |
| 16 | A |
| 17 | D |
| 18 | D |
| 19 | A |
| 20 | D |
| 21 | A |
| 22 | B |
| 23 | B |
| 24 | C |
| 25 | D |

**Tie Breaker 1 Answer**

A point is moving along the function $f(x) = \frac{1}{2}x^2$ such that $\frac{dx}{dt} = 3$ inches per second.




a) Find $\frac{dy}{dt}$ when $x = 5$

$$y = \frac{1}{2}x^2$$

$\frac{dy}{dt} = x\ \frac{dx}{dt}$, evaluated with $x = 5$ and $\frac{dx}{dt} = 3\ \frac{in}{sec}$,

$$\frac{dy}{dt} = 5\left(3\frac{in}{sec}\right) = 15\frac{in}{sec}$$

b) A triangle is formed when connecting the moving point to the origin and vertically to the $x$ axis as shown above. Find the rate of change of the area of the triangle when $x = 5$.

$$A = \frac{1}{2}bh = \frac{1}{2}(x)\left(\frac{1}{2}x^2\right) = \frac{1}{4}x^3$$

$\frac{dA}{dt} = \frac{3}{4}x^2\ \frac{dx}{dt}$, evaluated with $x = 5$ and $\frac{dx}{dt} = 3$ ,

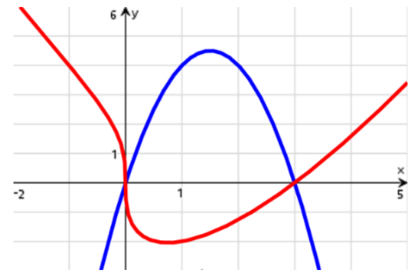
$$\frac{dA}{dt} = \frac{3}{4}(5)^2(3) = \frac{225}{4}\frac{in^2}{sec} \quad \text{or} \quad 56.25\frac{in^2}{sec}$$

Note: The units feel weird her because the y-value at $\frac{1}{2}x^2$ would be inches not square inches.

So, $\frac{3}{4}(5)^2(3)$ feels like it would have the units $\frac{in^3}{sec}$ instead of the actual $\frac{in^2}{sec}$

**Tie Breaker 2 Answer**

Use the functions $f(x) = 2x(3-x)$ and $g(x) = \sqrt[3]{x}(x-3)$ to answer the following problems. Write the integral used to find your answers. Round your answers to the nearest thousandths.




a) Find the area bounded by the curves.

$$\int_0^3 (f(x) - g(x))dx = \int_0^3 (2x(3-x) - \sqrt[3]{x}(x-3))dx$$

$\approx 13.172$

b) Find the volume of the solid when rotated around the line $y = 6$.

$$\pi \int_0^3 \left((6-g(x))^2 - (6-f(x))^2\right) dx$$

$$= \pi \int_0^3 \left((6 - \sqrt[3]{x}(x-3))^2 - (6 - 2x(3-x))^2\right) dx$$

$\approx 416.447$

c) Find the volume of the solid when rotated around the line $x = -1$

Shell method:

$$2\pi \int_0^3 (r(x)h(x))dx$$

$$2\pi \int_0^3 (x-(-1))(f(x) - g(x))dx = 2\pi \int_0^3 (x+1)(2x(3-x) - \sqrt[3]{x}(x-3))dx$$

$\approx 199.044$

d) Find the volume of the solid having the bounded area as its base and having rectangular cross sections with the height twice the base when the cross sections are cut perpendicular to the $x$-axis.

$$\int_0^3 (b \cdot h)\, dx = \int_0^3 \left((f(x) - g(x))(2)(f(x) - g(x))\right) dx = \int_0^3 \left((2)(f(x) - g(x))^2\right) dx$$

$$= \int_0^3 \left((2)(2x(3-x) - \sqrt[3]{x}(x-3))^2\right) dx$$

$\approx 134.044$

**Tie Breaker 3 Answer**

Consider the function $f(x) = x\ln(x^2+1)$.

a) Write the first four (4) non-zero terms and general term of the Maclaurin series for $g(x) = \ln(x+1)$.

Many may be able to write this from memory. The following is need if using derivatives and Taylor's theorem.
$g(x) = \ln(x+1), g'(x) = (x+1)^{-1}, g''(0) = -1(x+1)^{-2},\ g'''(0) = 2(x+1)^{-3}$, and $g^{iv}(x) = -3!\,(x+1)^{-4}$
$g(0) = 0, g'(0) = 1, g''(0) = -1, g'''(0)\ =\ 2!$, and $g^{iv}(0) = 3!$.

$$\ln(x+1) =\ 0 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{2!\,x^3}{3!} - \frac{3!\,x^4}{4!} + \cdots + \frac{(-1)^{n+1}(n-1)!\,x^3}{n!} + \cdots$$

$$\ln(x+1) =\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}\,x^n}{n} + \cdots$$

Note: Other equivalent general terms also accepted such as $\frac{(-1)^{n-1}\,x^n}{n}$ or $\frac{(-1)^n\,x^{n+1}}{n+1}$, etc.

b) Write the first four (4) non-zero terms and general term of the Maclaurin series for $h(x) = \ln(x^2+1)$.

Using that $h(x) = g(x^2)$, $\ln(x^2+1) =\ x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} +\ \cdots + \frac{(-1)^{n+1}x^{2n}}{n} +\ \cdots$

c) Find the interval of convergence of the Maclaurin series for $h(x) = \ln(x^2+1)$ .

Ratio Test: $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\frac{x^{2n+2}}{n+1}\cdot\frac{n}{x^{2n}}\right| = \lim\limits_{n\to\infty}\left|x^2\frac{n}{n+1}\right| = |x^2|\lim\limits_{n\to\infty}\left|\frac{n}{n+1}\right| = x^2.$

So, it converges when $x^2 < 1\ or\ -1 < x < 1$.

Checking the endpoints, both $x = -1\ and\ x = 1$ , $x^{2n} = 1$ yield the convergent alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. Therefore, the interval of convergence is $-1 \le x \le 1\ or\ [-1, 1]$.

d) Write the first four (4) non-zero terms and general term of the Maclaurin series for $f(x) = x\ln(x^2+1)$.

Using that $f(x) = x\cdot h(x)$, $x\ln(x^2+1) =\ x^3 - \frac{x^5}{2} + \frac{x^7}{3} - \frac{x^9}{4} +\ \cdots + \frac{(-1)^{n+1}x^{2n+1}}{n} +\ \cdots$

e) Estimate $f'\left(\frac{1}{2}\right)$ using the 4 terms found on part (d). What is the error between his approximation and $f'\left(\frac{1}{2}\right)$?

$$\frac{d}{dx}\left(x^3 - \frac{x^5}{2} + \frac{x^7}{3} - \frac{x^9}{4}\right) = 3x^2 - \frac{5x^4}{2} + \frac{7x^6}{3} - \frac{9x^8}{4}, f'\!\left(\tfrac{1}{2}\right) \approx 3\left(\frac{1}{2}\right)^2 - \frac{5\left(\frac{1}{2}\right)^4}{2} + \frac{7\left(\frac{1}{2}\right)^6}{3} - \frac{9\left(\frac{1}{2}\right)^8}{4} = \frac{323}{512}$$

$f'(x) = \ln(x^2+1) + \frac{x^2}{x^2+1}$ and $f'\left(\frac{1}{2}\right) = \ln\left(\frac{5}{4}\right) + \frac{2}{5}$. **Error =** $\left|\ln\left(\frac{5}{4}\right) + \frac{2}{5} - \frac{323}{512}\right| \approx |0.6231 - 0.6309| \approx 0.008$

Note: Finding $f'\left(\frac{1}{2}\right)$ with only the calculator is ok. Only the decimal appoximation of $f'\left(\frac{1}{2}\right)$ is also ok.