Arkansas Council of Teachers of Mathematics **Name ____________________________**

2023 Regional Calculus Competition

Work the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then, attempt the tiebreaker questions at the end starting with Tie Breaker #1, then #2, and finally #3. Turn in your answer sheet and the tiebreaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Figures aren't necessarily drawn to scale. Angles are given in radians unless otherwise stated. Assume all values are real.

1. $\lim_{x\to-\infty} \frac{\sqrt{16x^2+7}}{5x-4} =$

   A. $\frac{4}{5}$
   B. $-\frac{4}{5}$
   C. $\frac{7}{4}$
   D. $-\frac{7}{4}$
   E. $\frac{16}{5}$

2. Let f be a function that is continuous on the closed interval $[1,4]$ with $f(1) = 15$ and $f(4) = 5$. Which of the following statements must be true?

   A. $f'(x)$ is negative on $(1,4)$.
   B. $5 \leq f(3) \leq 15$
   C. $f'(x) = -\frac{10}{3}$ has at least one solution in the interval $(1,4)$.
   D. $f(x) = 8$ has at least one solution in the interval $[1,4]$.
   E. All of the above

3. Let $f(x)$ and $g(x)$ be continuous except for $x \neq 5$. Which of the following indicates that the graph of
   $y = \frac{f(x)}{g(x)}$ has a vertical asymptote?

   A. $\lim_{x\to5} \frac{f(x)}{g(x)}$ does not exist
   B. $g(5) = 0$
   C. $\lim_{x\to5^+} \frac{f(x)}{g(x)} = \infty$
   D. Both A and B
   E. A, B, and C

4. What is the average rate of change of $f(x) = 2x^3 + x$ on the interval $[-2,2]$?

   A. 9
   B. 25
   C. 7
   D. 16
   E. None of the above

5. Let $f(x)$ be continuous for all real numbers. Estimate the value of $f'(2.3)$ given the following table of values:

x	1.4	1.8	2.2	2.4	7
$f(x)$	-5	1	4	7	13

   A. $\frac{3}{2}$
   B. 15
   C. $\frac{11}{2}$
   D. $\frac{1}{15}$
   E. Not enough information given

6. If the position of an object at time t is $s(t) = t^3 - \frac{3}{2}t^2 - 6t + 1$, find the time where the velocity is a minimum.

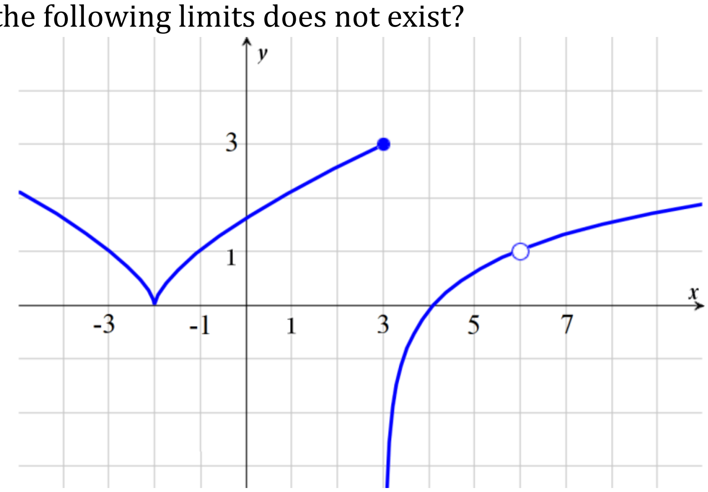
   A. $t = 1$
   B. $t = 2$
   C. $t = \ -1\ \&\ 2$
   D. $t = \frac{1}{2}$
   E. Not enough information; a closed interval is needed

7. $\lim\limits_{x \to \frac{\pi}{8}} \left(\frac{\tan(2x) - 1}{x - \frac{\pi}{8}} \right) =$

   A. 4
   B. $2\sqrt{2}$
   C. $\frac{\sqrt{2}}{2}$
   D. $2\sqrt{2}$
   E. Does not exist

8. The graph of $g(x)$ is shown below. Which of the following limits does not exist?

   A. $\lim\limits_{x \to -2} g(x)$
   B. $\lim\limits_{x \to 3^-} g(x)$
   C. $\lim\limits_{x \to 3} g(x)$
   D. $\lim\limits_{x \to 6} g(x)$
   E. A, C, & D



9. Find the x-value(s) where the function $f(x) = x^3 - 9x + 7$ has tangent lines that are parallel to $y = 3x - 7$.
   A. $\sqrt{3}$
   B. $\pm\sqrt{3}$
   C. 2
   D. ± 2
   E. None: $f(x)$ contains no tangent lines parallel to $y = 3x - 7$.

10. Find the horizontal asymptote(s) on $f(x) = \frac{4}{e^{3x}+2}$
   A. $y = 4$
   B. $y = 0$
   C. $y = 2$
   D. $y = \frac{4}{3}$
   E. $y = 0, 2$

11. Given $f(x) = x^3 - 6x$, find the value(s) of c guaranteed by the Mean Value Theorem on the interval $[-1, 2]$.
   A. $c = 1$
   B. $c = \frac{3}{2}$
   C. $c = \pm 1$
   D. $c = -\frac{3}{2}$
   E. None: MVT does not apply on this interval.

12. The position, in meters, of an object is given by $s(t) = t^4 - 6t^2 + 2t$ at t seconds. What is the average acceleration from $t = 1$ to $t = 3$?
   A. $90 \frac{m}{s^2}$
   B. $48 \frac{m}{s^2}$
   C. $40 \frac{m}{s^2}$
   D. $36 \frac{m}{s^2}$
   E. $18 \frac{m}{s^2}$

13. The derivative of the function f is continuous on the closed interval $[0, 4]$. Values of f and f' for selected values of x are given in the table below. If $\int_1^4 f'(t)\,dt = 8$ then $f(4) = ?$
   A. 3
   B. 5
   C. 7
   D. 10
   E. Not enough information given

x	0	1	2	3
$f(x)$	5	2	5	3
$f'(x)$	-3	1	3	4

14. Determine the slope of $x^2y^3 + 2y^4 = 6$ at the point $(2, 1)$.
    - A. $\frac{3}{10}$
    - B. $-\frac{1}{5}$
    - C. $\frac{1}{10}$
    - D. 0
    - E. None of the above

15. Consider the function defined by $f(x) = x^3 + x - 5$. If $g(x) = f^{-1}(x)$ and $f(2) = 5$, what is the value of $g'(5)$?
    - A. $\frac{1}{2}$
    - B. 2
    - C. $\frac{1}{5}$
    - D. $\frac{1}{13}$
    - E. Not enough information given

16. $\lim_{h \to 0} \frac{\sin^4(x+h) - \sin^4(x)}{h} =$
    - A. $\sin^4(x)$
    - B. $4\cos(x)\sin^3(x)$
    - C. $4\cos^3(x)$
    - D. 0
    - E. Does not exist

17. Find the following indefinite integral: $\int \left(x \cos(x^2)\right) dx =$
    - A. $-2\sin(x^2) + C$
    - B. $2\sin(x^2) + C$
    - C. $\frac{1}{2}\sin(x^2) + C$
    - D. $\frac{1}{2}x^2 \sin\left(\frac{1}{3}x^3\right) + C$
    - E. None of the above

18. A point is moving along the graph of the function $y = x^2 + 6x$ such that $dx/dt = 3$ centimeters per second. Find dy/dt when $y = 7$.
    - A. $\frac{dy}{dt} = 8\ \frac{cm}{s}$
    - B. $\frac{dy}{dt} = \pm 8\ \frac{cm}{s}$
    - C. $\frac{dy}{dt} = 20\ \frac{cm}{s}$
    - D. $\frac{dy}{dt} = 24\ \frac{cm}{s}$
    - E. $\frac{dy}{dt} = \pm 24\ \frac{cm}{s}$

19. Find the area bounded by $f(x) = x^2 + 1$ and $g(x) = 1 + 6x - x^2$.
    A. 9
    B. 12
    C. 18
    D. 21
    E. 36

20. Find the perimeter of the area bounded by $f(x) = x^2 + 1$ and $g(x) = 1 + 6x - x^2$.
    A. 9.747
    B. 14.598
    C. 17.813
    D. 18.173
    E. 19.494

21. Use the table below to approximate $\int_{-2}^{14} g(x)\, dx$. Use a trapezoidal approximation with 4 equal subintervals.

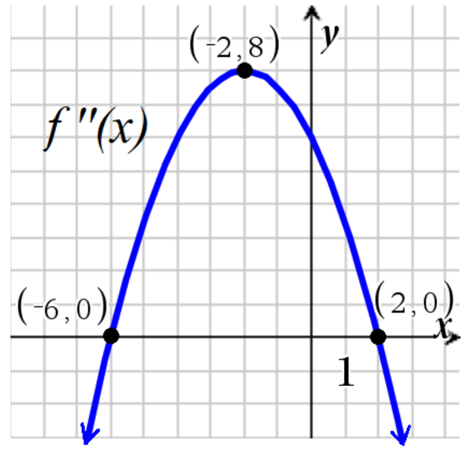
x	-2	0	2	4	6	8	10	12	14
$f(x)$	8	13	7	6	9	-3	-7	-10	-2

    A. 24
    B. 36
    C. 48
    D. 68
    E. None of the above

22. The position of a particle moving along a line is given by $s(t) = \frac{1}{3}t^3 - 4t^2 + 15t - 2023$ for $t \geq 0$. For what values of t is the speed of the particle increasing?
    A. $t > 4$
    B. $3 < t < 4$
    C. $t > 5$
    D. $0 < t < 3$ and $t > 5$
    E. $3 < t < 4$ and $t > 5$

 Name ______________________

23. Which of the following must be true based on the given graph of $f''(x)$ graphed to the right?

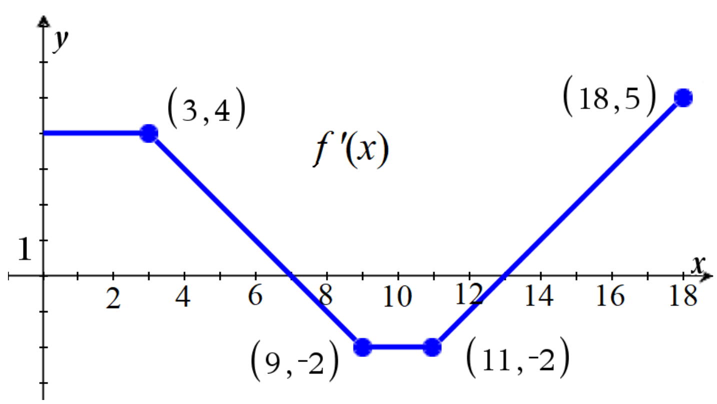


A. $f(x)$ is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.
B. $f(x)$ has inflection points at $x = -6$ and $x = 2$.
C. $f(x)$ has a relative max at $x = -2$.
D. $f(x)$ has a relative min at $x = -6$ and a relative max $x = 2$.
E. Both A and C

24. A 30' ladder is leaning against a wall and the base is on level ground. If the base is being pulled away from the wall at 3 ft/sec, at what rate is the top of the ladder sliding down the wall when it is 18 ft from the ground?

A. $-4\ \frac{ft}{sec}$
B. $-\frac{9}{4}\ \frac{ft}{sec}$
C. $24\ \frac{ft}{sec}$
D. $-6\ \frac{ft}{sec}$
E. $\frac{9}{4}\ \frac{ft}{sec}$

25. The graph of $y = f'(x)$ is graphed to the right.
If $f(13) = 20$, what is the value of $f(1)$?

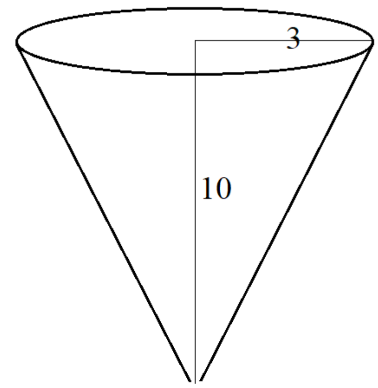


A. -8
B. 4
C. 8
D. 12
E. 28

Tie Breaker #1

Name: __________________

School: _________________



Conical volume: The volume of a cone is $V = \frac{1}{3}\pi\, r^2 h$.

A conical funnel is being used to put oil into your car. The 10" tall funnel has a 3" radius. The funnel is almost full of oil and the oil is flowing out at a rate of 2 cubic inches per second.

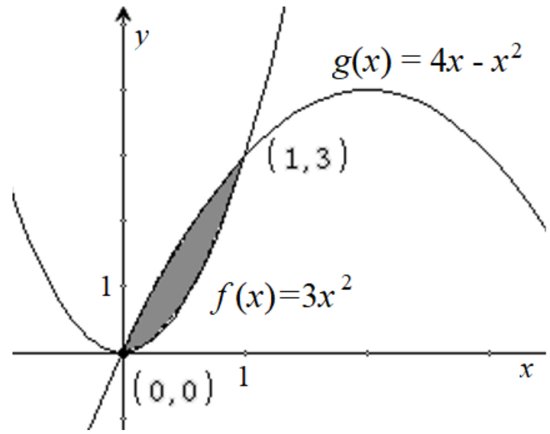
How fast is the height of oil in the funnel changing when there is only 5" left in the funnel?

Tie Breaker #2

Name: ___________________

School: _________________

Use the functions $f(x) = 3x^2$ and $g(x) = 4x - x^2$ (graphed on the right) to answer the following questions.



a) Let the shaded area bound by the graphs be the base of a fixed solid. What is the volume of the solid if cross sections, cut perpendicular to the x-axis, are semicircles?

b) Find the volume carved out in space if the bounded area is rotated around the line $x = 2$.

Tie Breaker #3

Name: ____________________

School: ___________________

For $0 \le t \le 24$ hours, the water level in a bay is given by the function W that satisfies the differential equation

$$\frac{dW}{dt} = \frac{2\sin\left(\frac{\pi}{2}t\right)}{3W}$$

Define $W(t)$ measures the depth of water by a dock in meters, and t is measured in hours. At time $t = 1$ hours, the depth of the water is 4 meters.

a) Write an equation for the line tangent to $y = W(t)$ when $t = 1$.

b) Find $y = W(t)$, the particular solution to the differential equation, given that $W(1) = 4$.

Solutions to 2023 Regional Calculus Competition

#	Ans.	Description
1	B	$\lim_{x\to-\infty}\frac{\sqrt{16x^2+7}}{5x-4} = \lim_{x\to-\infty}\left(\sqrt{16+\frac{7}{x^2}}/\left(-5-\frac{4}{\lvert x\rvert}\right)\right) = \frac{\sqrt{16}}{-5} = -\frac{4}{5}$. Note: $\sqrt{x^2} = \lvert x\rvert$ and $\frac{5x}{\lvert x\rvert} = -5$ because $x < 0$.
2	D	D: Intermediate Value Theorem (Choice C looks plausible but without differentiability MVT does not apply.)
3	C	A could be a jump disc etc. B: could have a hole C: Must have a vertical asymptote.
4	A	Average rate, $\frac{\Delta y}{\Delta x} = \frac{36}{4} = 9$
5	B	Use the average rate of change from $x = 2.2$ to $x = 2.4$ to approximate $f'(2.3)$. $\frac{\Delta y}{\Delta x} = \frac{7-3}{2.4-2.2} = 15$
6	D	$v'(t) = a(t) = 6t - 3$ changes from – to + at $t = 1/2$ (only crit val) indicating $v(t)$ is a min. **(Corrected)**
7	A	L'Hôpital's Rule Applies. $\lim_{x\to\pi/8}\frac{2sec^2(2x)}{1} = 2\left(sec\frac{\pi}{4}\right)^2 = 2\left(\sqrt{2}\right)^2 = 4$
8	C	Only at $x = 3$ does $\lim_{x\to 3^-} g(x) \neq \lim_{x\to 3^+} g(x)$
9	D	$f'(x) = 3\,, 3x^2 - 9 = 3, x^2 = 4, x = \pm 2$
10	E	Since $\lim_{x\to-\infty} e^{3x} = 0$, $\lim_{x\to-\infty} f(x) = \frac{4}{2} = 2$ and since $\lim_{x\to\infty} e^{3x} = \infty$, $\lim_{x\to\infty} f(x) = 0$
11	A	$\frac{\Delta y}{\Delta x}$ on $[-1, 2] = -3, f'(x) = 3x^2 - 6 = -3, x = \pm 1$ but MVT only guaranteed on $(-1,2)$; $c \neq -1$
12	C	Average Acceleration, as an average rate of change, $= \frac{\Delta v}{\Delta t} = \frac{v(3)-v(1)}{3-1} = 40$
13	D	Fundamental Theorem of Calculus: $\int_1^4 f'(t)dt = 8, f(4) - f(1) = 8$ $f(4) = 8 + f(1) = 8 + 2 = 10$

#	Ans.	Description
14	B	Imp Diff: $2xy^3 + 3x^2xy^2\frac{dy}{dx} + 8y^3\frac{dy}{dx} = 0, @\ (2,1): 4 + 12\frac{dy}{dx} + 8\frac{dy}{dx} = 0,$ $$20\frac{dy}{dx} = -4, \frac{dy}{dx} = -\frac{1}{5}$$
15	D	Can switch x, y and use Implicit Differentiation, or $$g'(5) = \frac{1}{f'\big(g(5)\big)} = \frac{1}{f'(2)} = \frac{1}{3(4)+1} = \frac{1}{13}$$
16	B	Definition of the derivative where $f(x) = \sin^4 x, f'(x) = 4\sin^3(x) \cdot \cos(x)$
17	C	Use u-substitution or use understanding of the chain rule
18	E	Solve $x^2 + 6x = 7, x = -7,1$. Take the $\frac{d}{dt}$ to get $\frac{dy}{dt} = (2x+6)\frac{dx}{dt}$. With $\frac{dx}{dt} = 3$ and $x = -7,1, \frac{dy}{dt} = \pm 24$
19	A	$\int_0^3 (g(x) - f(x))dx = 9$
20	E	With top and bottom having the same length, perimeter = $2\int_0^3 \sqrt{1 + (f'(x))^2}dx \approx 19.494$
21	C	$$4\left(\frac{8+7}{2}\right) + 4\left(\frac{7+9}{2}\right) + 4\left(\frac{9-7}{2}\right) + 4\left(\frac{-7-2}{2}\right) = 48$$
22	E	Need Vel and Accel to be the same sign. Plot crit. values & check, Both neg: $3 < t < 4$ and both pos: $t > 5$
23	B	$f(x)$ has inflection points at $x = -6$ and $x = 2$ because $f''(x)$ changed signs.
24	A	$x^2 + y^2 = 30, 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0, \frac{dy}{dt} = -x\frac{dy}{dt}/y. x = 24$ when $y = 18$ and $\frac{dy}{dt} = \frac{-24(3)}{18} =$ $-4\ \frac{ft}{sec}$
25	D	$\int_{13}^{1} f'(x)dx = f(1) - f(13),$ $f(1) = f(13) + \int_{13}^{1} f'(x)dx = 20 - \int_{1}^{13} f'(x)dx = 20 - (16-8) = 12$

Tie Breaker 1 Answer

A conical funnel is being used to put oil into your car. The 10" tall funnel has a 3" radius. The funnel is almost full of oil and the oil is flowing out at a rate of 2 cubic inches per second.

How fast is the height of oil in the funnel changing when there is only 5" left in the funnel?

Given: $\frac{dV}{dt} = -2\frac{in^3}{s}$, $r = \frac{3}{10}h$. So, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{3}{10}h\right)^2 h = \frac{1}{3}\pi \frac{9}{100}h^3$

With $V = \frac{1}{3}\pi \frac{9}{100}h^3$, $\frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt}$, and $\frac{dh}{dt} = \frac{100}{9\pi h^2}\frac{dV}{dt}$

Now with $\frac{dV}{dt} = -2\frac{in^3}{s}$ *and* $h = 5in$, $\frac{dh}{dt} = \frac{100\left(-2\frac{in^3}{s}\right)}{9\pi(5in)^2} = -\boldsymbol{\frac{8}{9\pi}\frac{in}{s}}$

Tie Breaker 2 Answer

Use the functions $f(x) = 3x^2$ and $g(x) = 4x - 4x^2$ graphed at the right to answer the following questions.

a) Let the shaded area bound by the graphs be the base of a fixed solid. What is the volume of the solid if cross sections, cut perpendicular to the x-axis, are semicircles?
b) Find the volume carved out in space if the bounded area is rotated around the line $x = 2$.

a) Area of each cross section is $\frac{1}{2}\pi r^2$ where $r = \frac{g(x)-f(x)}{2}$. These infinite cross sections can be added up with
$\boldsymbol{\frac{1}{2}\pi\int_0^1\left(\frac{4x-x^2-3x^2}{2}\right)^2 dx = \frac{\pi}{15}}$. Work: $\frac{1}{2}\pi\int_0^1\left(\frac{4x-x^2-3x^2}{2}\right)^2 dx = \frac{\pi}{2}\int_0^1(2x-2x^2)^2\,dx$
$= \frac{\pi}{2}\int_0^1(4x^2-8x^3+4x^4)\,dx = \frac{\pi}{2}\left[\frac{4}{3}x^3 - 2x^4 + \frac{4}{5}x^5\right]_0^1 = \frac{\pi}{2}\left[\frac{4}{3} - 2 + \frac{4}{5} - 0\right] = \frac{\pi}{15}$.

b) Slice Vertically and use Shell method: $\boldsymbol{2\pi\int_0^1 r(x)h(x)\,dx = 2\pi\int_0^1((2-x)(4x-4x^2))\,dx = 2\pi}$
Or
Solve for x and slice horizontally and use washer method:

$f(x) = y = 3x^2 \xrightarrow{yields} x = +\sqrt{\frac{y}{3}}$ and $g(x) = y = 4x - 4x^2 \xrightarrow{yields} x = 2 - \sqrt{4-y}$ (Note: requires completing the square to solve for y and the choice of $-\sqrt{\ }$ for the left half of the parabola.

Volume via Washer: $\boldsymbol{\pi\int_0^3\left(\left(2-\left(2-\sqrt{4-y}\right)\right)^2 + \left(2-\sqrt{\frac{y}{3}}\right)^2\right)dy = \pi\int_0^3\left(4 - y + \left(2-\sqrt{\frac{y}{3}}\right)^2\right)dy = 2\pi}$

Tie Breaker 3 Answer

For $0 \leq t \leq 24$ hours, the water level in the bay is given by the function W that satisfies the differential equation $\frac{dW}{dt} = \frac{2sin\left(\frac{\pi}{2}t\right)}{3W}$. $W(t)$ measures the depth of water by a dock in meters, and t is measured in hours. At time $t = 1$ hours, the depth of the water is 4 meters.

a) *Write an equation for the line tangent to $y = W(t)$ when $t = 1$.*
b) *Find $y = W(t)$, the particular solution to the differential equation, given that $W(1) = 4$.*

a) $\frac{dW}{dt} = \frac{2sin\left(\frac{\pi}{2}t\right)}{3W}$, $\frac{dW}{dt}$ @$(1,4) = \frac{2\,(1)}{3(4)} = \frac{1}{6}$. So, an equation of the line at $(1, 4)$ with slope $\frac{1}{6}$ is $y - 4 = \frac{1}{6}(t-1)$ or $y = \frac{1}{6}t + \frac{23}{6}$

b) Separate the variables and integrate: $3W\ dW = 2sin\left(\frac{\pi}{2}t\right)dt$

$$\int 3W\ dW = \int 2sin\left(\frac{\pi}{2}t\right)dt$$

Must have the constant of integration: $\frac{3}{2}W^2 = -\frac{4}{\pi}cos\left(\frac{\pi}{2}t\right) + C$

Use the initial condition to solve for C: $\frac{3}{2}(4)^2 = -\frac{4}{\pi}cos\left(\frac{\pi}{2}\right) + C$

$$24 = C$$

$$\frac{3}{2}W^2 = -\frac{4}{\pi}cos\left(\frac{\pi}{2}t\right) + 24$$

$$W^2 = -\frac{8}{3\pi}cos\left(\frac{\pi}{2}t\right) + 16$$

Choose $+\sqrt{\quad}$ because of $(1, 4)$ $W(t) = \sqrt{-\frac{8}{3\pi}cos\left(\frac{\pi}{2}t\right) + 16}$, $0 \leq t \leq 24$.