

ACTM State Geometry Competition 2017

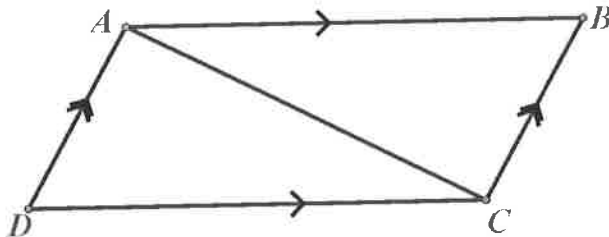
Begin by removing the three tie breaker sheets at the end of the exam and writing your name on all three pages. Work the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Make sure you attempt the tie-breaker questions at the end of the test starting with tie breaker 1, then 2, and then 3 if you have time. Turn in your answer sheet and the tie breaker pages when you are finished. You may keep the pages with the multiple-choice questions. Choose the single *best* (most completely correct) response for each of the multiple-choice questions below.

Notations and Definitions:

- All questions on this test are in **Euclidean Geometry** unless indicated otherwise.
 - All angles are measured in **radians**. π radians = 180° .
 - AB indicates the distance between points A and B .
 - $A-B-C$ indicates that B is *between* A and C that is: A , B , and C are collinear and $AB + BC = AC$.
 - A *cyclic polygon* is a polygon for which there is a single *circumscribed* circle containing all the vertices of the polygon.
 - A *kite* is a quadrilateral with at least two non-overlapping pairs of congruent consecutive sides. Its *major diagonal* has endpoints where the congruent sides meet.
 - An *isometry* (rigid transformation) is a transformation mapping every preimage to a congruent image.
 - **Z Property:** Alternate interior angles formed by a transversal to lines l and m are congruent if and only if l and m are parallel.
1. Every isometry can be formed by the composition of at most ____ reflections.
 - A. 1
 - B. 2
 - C. 3
 - D. 4
 - E. Each of the other answer is incorrect.
 2. The composition of two reflections about intersecting lines is a single _____.
 - A. Rotation
 - B. Reflection
 - C. Translation
 - D. Glide-Reflection
 - E. Dilation
 3. Which of the following is not an isometry?
 - A. Rotation
 - B. Reflection
 - C. Translation
 - D. Glide-Reflection
 - E. Dilation

4. What is the inverse of a translation by vector $\langle 3, 2 \rangle$?
- A. A translation by vector $\langle 3, 2 \rangle$
 - B. A translation by vector $\langle 2, 3 \rangle$
 - C. A translation by vector $\langle -3, -2 \rangle$
 - D. A translation by vector $\langle -2, -3 \rangle$
 - E. Each of the other answers is incorrect.
5. If a circle can be *inscribed* in a polygon (i.e. tangent to each side), then the center of the inscribed circle is at the common intersection of the ____.
- A. bisectors of the interior angles
 - B. perpendicular bisectors of the sides
 - C. diagonals
 - D. medians
 - E. Each of the other answers is incorrect.
6. For any cyclic polygon, the center of the *circumscribed* circle (containing the vertices) is at the common intersection of the ____.
- A. bisectors of the interior angles
 - B. perpendicular bisectors of the sides
 - C. diagonals
 - D. medians
 - E. Each of the other answers is incorrect.
7. Which of the following types of polygons is always cyclic?
- A. Triangle
 - B. Quadrilaterals
 - C. Pentagons
 - D. Hexagons
 - E. Each of the other answers is incorrect.
8. For any cyclic quadrilateral, each pair of opposite interior angles are ____.
- A. Congruent
 - B. Complementary
 - C. Supplementary
 - D. Equal
 - E. Each of the other answers is incorrect.
9. A trapezoid with a pair of supplementary opposite interior angles is ____.
- A. a Right Trapezoid
 - B. a Rectangle
 - C. an Isosceles Trapezoid
 - D. a Square
 - E. Each of the other answers is incorrect.

10. Given the following parallelogram and one of its diagonals:



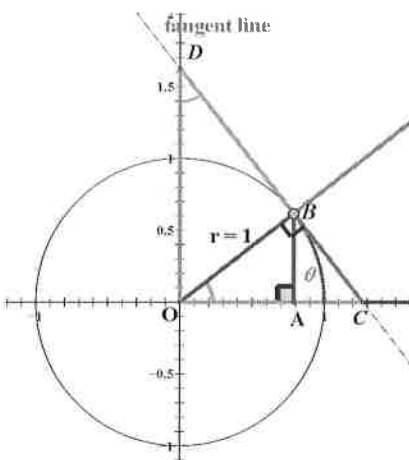
Starting from the definition of a parallelogram, which of the following is an outline of a correct deduction based only on the given information?

- A. $\angle DCA \cong \angle BAC$ and $\angle BCA \cong \angle DAC$ by the Z Property and $\triangle ABC \cong \triangle CDA$ by ASA Triangle Congruence Theorem
 - B. $\angle DCA \cong \angle BAC$ and $\angle BCA \cong \angle DAC$ by the Z Property and $\triangle ABC \cong \triangle ADC$ by ASA Triangle Congruence Theorem
 - C. $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$ and $\triangle ABC \cong \triangle CDA$ by SSS Triangle Congruence Theorem
 - D. $\angle DCA \cong \angle BAC$ and $\angle BCA \cong \angle DAC$ by the Z Property and $\triangle ABC \cong \triangle ADC$ by AAS Triangle Congruence Theorem
 - E. Each of the other answers is incorrect.
11. The sum of the measures of the interior angles of a convex n -gon is ____.
- A. $(n-2)\frac{\pi}{2}$
 - B. $n\pi$
 - C. $n\pi - \pi$
 - D. $(n-2)\pi$
 - E. Each of the other answers is incorrect.
12. Which of the following is **not** a *minimal* set of conditions for a quadrilateral to be a rectangle?
- A. It has four right interior angles.
 - B. It is equiangular.
 - C. It is a parallelogram containing a right interior angle.
 - D. It is an isosceles trapezoid with a right interior angle.
 - E. Each pair of its opposite interior angles are congruent and it has a right interior angle.
13. Which of the following types of quadrilaterals does not necessarily have a pair of congruent angles?
- A. Rectangles
 - B. Kites
 - C. Parallelograms
 - D. Isosceles Trapezoids
 - E. Trapezoids
14. Which of the following is NOT a property of an Isosceles Triangle?
- A. It has a pair of congruent angles.
 - B. It has a pair of sides which are congruent.
 - C. Each of its medians is also a bisector of an interior angle.
 - D. The sum of the lengths of any two of its sides is greater than the length of the third side.
 - E. The perpendicular bisector of one of its sides is a bisector of an interior angle.

15. If we construct the major diagonal of a kite, then we can immediately show from the definition that the two triangles formed are:
- Congruent by the SAS Triangle Congruence Postulate
 - Congruent by the SSS Triangle Congruence Theorem
 - Congruent by the ASA Triangle Congruence Theorem
 - Not necessarily congruent.
 - Each of the other answers is incorrect.
16. How many different (non-congruent) parallelograms can be constructed with sides of length 4 cm, 5 cm, 4cm, and 5 cm?
- 0
 - Exactly 1
 - Exactly 2
 - Infinitely many
 - It cannot be determined from the information given.
17. A quadrilateral in which both pair of opposite angles are congruent must be a _____.
- Rhombus
 - Kite
 - Parallelogram
 - Rectangle
 - Isosceles Trapezoid
18. Which of the following is NOT a valid proposition concerning similarity of triangles in Euclidean Geometry?
- Given a correspondence between two right triangles, if the lengths of a pair of corresponding legs are proportional and the lengths of corresponding hypotenuses are proportional then the correspondence is a similarity.
 - Given a correspondence between two triangles, if the lengths of two pairs of corresponding sides are proportional and the angles opposite one of these sides are congruent, then the correspondence is a similarity.
 - Given a correspondence between two triangles, if the lengths of all of the corresponding sides are proportional, then the correspondence is a similarity.
 - Given a correspondence between two triangles, if two of the corresponding angles are congruent, then the correspondence is a similarity.
 - Given any right triangle, the three triangles contained in the union of the given triangle and the altitude to the hypotenuse are similar.
19. Given $\triangle ABC$, $A-B-D$, $m\angle DBC = \frac{\pi}{3}$, and $m\angle BAC = \frac{\pi}{6}$, what is $m\angle ACB$?
- $\frac{2\pi}{3}$
 - $\frac{4\pi}{3}$
 - $\frac{\pi}{6}$
 - $\frac{\pi}{3}$
 - Each of the other answers is incorrect.

20. Given a circle X with chords \overline{AB} and \overline{CD} intersecting at point E , $m\angle AC = \frac{\pi}{8}$, and $m\angle DB = \frac{\pi}{4}$ then $m\angle AEC =$
- A. $\frac{3\pi}{8}$
 - B. $\frac{\pi}{16}$
 - C. $\frac{3\pi}{16}$
 - D. $\frac{\pi}{8}$
 - E. Each of the other answers is incorrect.
21. If the measures of the angles of a triangle are x , y , and $x + y$ for some positive real numbers x and y , then the triangle is:
- A. Acute
 - B. Right
 - C. Obtuse
 - D. Isosceles
 - E. Each of the other answers is incorrect.
22. Given two triangles such that two pairs of corresponding sides are congruent and the pair of corresponding angles opposite one of those sides is congruent (SSA). Which of the following statements is NOT true?
- A. The two triangles are congruent if both of the given triangles are acute.
 - B. The two triangles are congruent if the congruent angles are right.
 - C. If the two triangles are not congruent then remaining pair of angles not included by the given congruent sides are not congruent but are supplementary.
 - D. If it is known that both triangles are obtuse then the two triangles are congruent.
 - E. The two triangles are congruent if in one triangle the side opposite the congruent angle is larger than the other side given to be congruent to a side in the other triangle.

For Questions 23-25: In the diagram below we see an acute angle, $\angle BOA$, of measure θ which is placed on a coordinate grid so that O is the origin and \overrightarrow{OA} is the positive portion of the x -axis. A unit circle of radius 1 is constructed with center O , and this circle contains point B . Note that arclength of the arc intercepted by this central angle is the same as the radian measure of the angle, θ . We also construct a line tangent to the circle at point B . The intersection of this tangent line with the x and y axes are the points C and D , respectively. Finally, point A is the intersection of the line perpendicular to the x -axis through point B . The value of each of the six basic trigonometric functions of the angle measuring θ can be seen as a length in the diagram.



23. Which of the lengths represents $\cos(\theta)$?
 - A. OA
 - B. AB
 - C. BC
 - D. OC
 - E. BD
24. Which of the lengths represents $\tan(\theta)$?
 - A. OA
 - B. AB
 - C. BC
 - D. OC
 - E. BD
25. Which of the lengths represents $\cot(\theta) = \frac{1}{\tan(\theta)}$?
 - A. OA
 - B. AB
 - C. BC
 - D. OC
 - E. BD

Name _____

Tie Breaker 1:

Prove that the major diagonal of a convex kite perpendicularly bisects the minor diagonal. Make your argument starting with the definition and do not use any other properties of the kite which you do not prove.

Name _____

Tie Breaker 2:

Prove the converse of the Pythagorean Theorem: If we are given a triangle with sides of lengths a , b , and c where $c^2 = a^2 + b^2$ then the angle opposite the side of length c is a right angle.

Note: You may assume that the Pythagorean Theorem has already been proved.

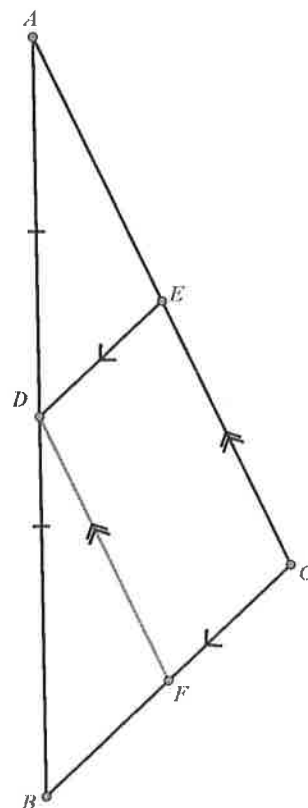
Name _____

Tie Breaker 3:

Given: There exists $\triangle ABC$. D is the midpoint of \overline{AB} . E is a point between A and C . $\overline{DE} \parallel \overline{BC}$.

Prove: $2(DE) = BC$, $\overline{AE} \cong \overline{EC}$, and $\triangle ADE \sim \triangle ABC$.

(Note: You *cannot* use the AA Triangle Similarity Theorem since this result is an important step used in the proof of the AA Triangle Similarity Theorem. Hint: Construct the pictured line segment parallel to \overline{AC} .)



**ACTM State Geometry Competition
2017
Key**

1.	C
2.	A
3.	E
4.	C
5.	A
6.	B
7.	A
8.	C
9.	C
10.	A
11.	D
12.	A
13.	E
14.	C
15.	B
16.	D
17.	C
18.	B
19.	C
20.	C
21.	B
22.	D
23.	A
24.	C
25.	E

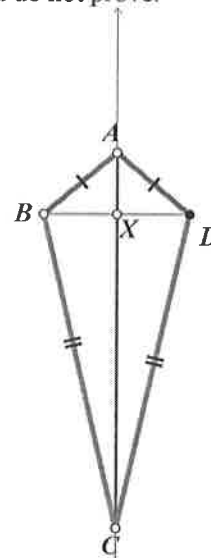
Name Key

Tie Breaker 1:

Prove that the major diagonal of a convex kite perpendicularly bisects the minor diagonal. Make your argument starting with the definition and do not use any other properties of the kite which you do not prove.

Proof:

We are given a kite $ABCD$. By definition, the kite must have two disjoint pairs of congruent sides so without loss of generality let $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$. Given any two distinct points there exists a line segment with those two points as endpoints. Construct \overline{AC} , which is, by definition, the major diagonal of kite $ABCD$. Note that every line segment is congruent to itself so $\overline{AC} \cong \overline{AC}$, and we already have $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$, so by the SSS Triangle Congruence Theorem $\triangle ABC \cong \triangle ADC$. By definition, corresponding parts of congruent triangles are congruent so $\angle ABC \cong \angle ADC$. (So, the major diagonal bisects the angle A .)



Given two distinct points there exists a unique line segment with those two points as endpoints, so we can construct \overline{BD} , which, by definition, is the minor diagonal. Since kite $ABCD$ is convex the diagonals must intersect. Let X be the point of intersection of \overline{AC} and \overline{BD} .

Note that, by definition, $\triangle ABD$ is isosceles and thus $\angle ABD \cong \angle ADB$ by the Isosceles Triangle Theorem. Since $\angle ABX = \angle ABD \cong \angle ADB = \angle ADX$, $\overline{AB} \cong \overline{AD}$, and $\angle ABX = \angle ABD \cong \angle ADB = \angle ADX$ we see that by the ASA Triangle Congruence Theorem $\triangle ABX \cong \triangle ADX$. By definition, corresponding parts of congruent triangles are congruent so $\overline{BX} \cong \overline{DX}$ and $\angle AXB \cong \angle AXD$. However, these two angles also form a linear pair and are thus congruent. So, we have $\angle AXB$ and $\angle AXD$ are both right angles. By definition \overline{AC} perpendicularly bisects \overline{BD} .

Therefore, we have shown that the major diagonal of a convex kite perpendicularly bisects the minor diagonal. ■

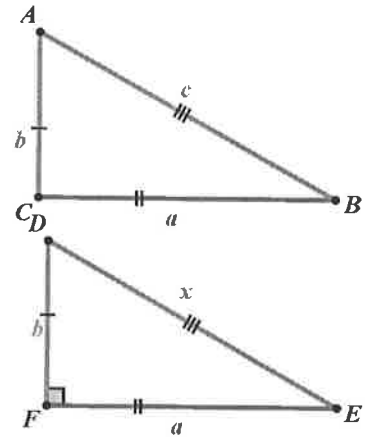
Tie Breaker 2:

Prove the converse of the Pythagorean Theorem: If we are given a triangle with sides of lengths a , b , and c where $c^2 = a^2 + b^2$ then the angle opposite the side of length c is a right angle.

Note: You may assume that the Pythagorean Theorem has already been proved.

Proof:

We are given a triangle with sides of lengths a , b , and c where $c^2 = a^2 + b^2$. Construct a right angle with vertex at point F . Let D be a point on one ray of this angle with $FD = b$, and let E be a point on the other ray of the angle so that $FE = a$. By the Pythagorean Theorem (proved independently) we can conclude that $DE = x$ satisfies $a^2 + b^2 = x^2$. However, we also know that $a^2 + b^2 = c^2$, and thus, $x^2 = c^2$. a , b , c , and x are all positive values so $x = c$. Now by the SSS Triangle Congruence Theorem we conclude that $\triangle ABC \cong \triangle DEF$. Corresponding parts of congruent triangles are congruent so $\angle ACB \cong \angle DFE$ and thus $\angle ACB$ is a right angle. This shows that if the side lengths of a triangle satisfy $a^2 + b^2 = c^2$ then the angle opposite the side of length c is a right angle. ■



Name _____

Tie Breaker 3:

Given: There exists $\triangle ABC$. D is the midpoint of \overline{AB} . E is a point between A and C . $\overline{DE} \parallel \overline{BC}$.

Prove: $2(DE) = BC$, $\overline{AE} \cong \overline{EC}$, and $\triangle ADE \sim \triangle ABC$.

(Note: You *cannot* use the AA Triangle Similarity Theorem since this result is an important step used in the proof of the AA Triangle Similarity Theorem. Hint: Construct the pictured line segment parallel to \overline{AC} .)

Proof:

We are given the following: There exists $\triangle ABC$. D is the midpoint of \overline{AB} . E is a point between A and C . $\overline{DE} \parallel \overline{BC}$.

Construct the unique line through D parallel to \overline{AC} and call its intersection with \overline{BC} point F .

Note that \overline{AB} is a transversal to parallel lines \overline{DE} and \overline{CB} . By the "F" Property corresponding angles thus constructed are congruent so $\angle ADE \cong \angle ABC$. Similarly, \overline{AB} is a transversal to parallel lines \overline{AC} and \overline{DF} . Therefore, by the "F" Property $\angle BAC \cong \angle BDF$. Similarly, \overline{AC} is a transversal to parallel lines \overline{DE} and \overline{CB} , thus $\angle AED \cong \angle ACB$.

Since we have $\angle BAC \cong \angle BDF$, $AD = DB$, and $\angle ADE \cong \angle ABC$, we can conclude by the ASA Triangle Congruence Theorem that $\triangle ADE \cong \triangle DBF$. Corresponding parts of congruent triangles are congruent so $DE = BF$ and $AE = DF$.

By definition, Quadrilateral $DECF$ is a parallelogram. By an earlier theorem opposite sides of a parallelogram are congruent. So, $EC = DF$ and $DE = CF$.

So, we see that $BF = FC$ and $AE = EC$ so F is the midpoint of \overline{BC} and E is the midpoint of \overline{AC} . Furthermore, we have $BC = 2 DE$, $AC = 2 AE$, and $AB = 2 AD$ so the sides of $\triangle ABC$ are proportional to the sides of $\triangle ADE$. Furthermore, we have shown that corresponding angles of these two triangles are congruent so, by definition, the two triangles are similar: $\triangle ABC \sim \triangle ADE$ with a proportionality factor of 2. ■

