Arkansas Council of Teachers of Mathematics
2020 Regional Calculus Exam

Work the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then, attempt the tiebreaker questions at the end starting with Tie Breaker #1, then #2, and finally #3. Turn in your answer sheet and the tiebreaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Figures aren't necessarily drawn to scale. Angles are given in radians unless otherwise stated. Assume all values are real.

1. A function $f(x)$ continuous on the interval $[-1, 4]$ has values according to the following table:

x	$f(x)$
-1	1
1	1
2	-1
3	1
4	1

Identify the interval(s) that contain a value c such that $f(c) = 0$.

A. On the interval $(-1, 1)$
B. On the interval $(1, 2)$
C. On the interval $(-1, 1) \cup (1, 2)$
D. On the interval $(1,2) \cup (2,3)$
E. Not enough information given

2. The *hyperbolic sine* function is defined as $sinh(x) = \frac{1}{2}(e^x - e^{-x})$. Evaluate $\frac{d}{dx}[sinh(x^2)]$.

A. $\frac{1}{2}(e^x + e^{-x})$
B. $\frac{1}{2}(e^x - e^{-x})$
C. $x\left(e^{x^2} + e^{-x^2}\right)$
D. $x\left(e^{x^2} - e^{-x^2}\right)$
E. Not possible to determine

3. Evaluate the following integral: $\int \frac{x}{e^x} dx$

A. $\frac{1}{e^x} + C$
B. $\frac{x^2}{e^x} + C$
C. $\frac{-x-1}{e^x} + C$
D. $\frac{-x+1}{e^x} + C$
E. Unable to determine

4. Find the exact value of the following limit:

$$\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} =$$

   A. $\sqrt{3}$
   B. 0
   C. $\frac{\sqrt{3}}{6}$
   D. 1
   E. The limit does not exist

5. Let $f(x) = \frac{5}{2}x^2 - e^x$. Find the value of x for which the second derivative $f''(x)$ equals zero.
   A. $\ln(5)$
   B. $5e$
   C. 0
   D. e^5
   E. None of the above

6. Find the derivative of $f(x) = \left(1 + x^4 - \frac{1}{x}\right)^{5/3}$.
   A. $\frac{20}{3}x^{17/3} + \frac{5}{3x^{8/3}}$
   B. $\frac{5}{3}\left(1 + x^4 - \frac{1}{x}\right)^{2/3}$
   C. $\frac{5}{3}\left(1 + x^4 - \frac{1}{x}\right)^{2/3}\left(4x^3 + \frac{1}{x^2}\right)$
   D. $\frac{5}{3}x^{2/3}\left(4x^3 + \frac{1}{x^2}\right)$
   E. None of the above

7. Use implicit differentiation to find an equation of the tangent line to the curve $\sin(x) + \cos(y) = 1$ at the coordinate $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.
   A. $y = \frac{\pi}{2}$
   B. $y = \pi$
   C. $y = x - \frac{\pi}{2}$
   D. $y = 4\left(x - \frac{\pi}{2}\right)$
   E. None of the above

8. A function has x and $f(x)$ values according to the following table. Use this data to calculate the right-hand Riemann Sum estimation of the area bounded between the curve $f(x)$, x-axis, $x = 0$, and $x = 8$, using a regular partition of four rectangles.

x	$f(x)$
0	1
2	5
4	17
6	37
8	65

   A. $A = 120\ units^2$
   B. $A = 178.\overline{6}\ units^2$
   C. $A = 248\ units^2$
   D. $A = 820\ units^2$
   E. None of the above

9. Consider the polynomial $f(x) = x^3 + kx^2 + 14$, where k is a real valued constant. Find restriction(s) for the real constant k so that $f(x)$ has exactly one inflection point.
   A. $k = -3x$
   B. $k = -\frac{3x}{2}$
   C. No values for k exist.
   D. k can be any real number
   E. Not enough information given

10. Find the volume of revolution if $y(x) = 2 + \frac{x^2}{9}$ is revolved about the x-axis between $x = 0$ and $x = 3$.
   A. $Volume = 7\ units^3$
   B. $Volume = 7\pi\ units^3$
   C. $Volume = \frac{45}{4}\pi\ units^3$
   D. $Volume = \frac{83}{5}\pi\ units^3$
   E. Not enough information given

11. Consider the piecewise function $f(x)$ given below. Is $f(x)$ continuous on the interval $-\infty < x < \infty$?

$$f(x) = \begin{cases} x^3 - 10 & x \leq 2 \\ -4x + 6 & 2 < x < 4 \\ -x^2 + 5 & 4 \leq x \end{cases}$$

A. The function is continuous for all x-values.
B. The function is continuous for all values of x except at $x = 2$.
C. The function is continuous for all values of x except at $x = 4$.
D. The function is continuous for all values of x except at $x = 2$ and $x = 4$.
E. Not able to determine

12. Evaluate the following limit:

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} =$$

A. $-\frac{\sqrt{2}}{3}$
B. $-\frac{2}{3}$
C. $\frac{\sqrt{2}}{3}$
D. $\frac{2}{3}$
E. Does Not Exist

13. The volume of a sphere is increasing at a rate of $9\pi \frac{cm^3}{sec}$. Find the rate of change of its radius when its volume is $36\pi\ cm^3$.

A. $\frac{1}{3}\frac{cm}{s}$
B. $\frac{1}{4}\frac{cm}{s}$
C. $3\frac{cm}{s}$
D. $4\frac{cm}{s}$
E. Unable to determine

14. Determine the derivative of the function $y = |x|$.

A. $y' = 0$
B. $y' = 1$
C. $y' = \frac{x}{|x|}$
D. $y' = \frac{\sqrt{x^2}}{x}$
E. Derivative does not exist

15. The average height of the function $y = x^2 + x$ between $x = 0$ and $x = 2$ is
   A. $\frac{14}{3}$
   B. $\frac{7}{3}$
   C. 2
   D. $\frac{1}{2}$
   E. Unable to Determine

16. Use logarithmic differentiation to find the derivative of $y = x^x$.
   A. $y' = 1^1$
   B. $y' = x \cdot \ln(x)$
   C. $y' = 1 + \ln(x)$
   D. $y' = x^x(1 + \ln(x))$
   E. No derivative exists.

17. Define $y = e^{rx}$. For what value(s) of r is y a solution to the differential equation $2y'' + y' - y = 0$?
   A. $r = -1$
   B. $r = \frac{1}{2}$
   C. $r = 1$
   D. $r = \frac{1}{2}$ and $r = -1$
   E. No values exist.

18. Evaluate the integral:

$$\int \sin(t) \cdot e^{\cos(t)}\, dt =$$

   A. $-e^{cos(t)} + c$
   B. $-e^{sin(t)} + c$
   C. $-cos(t) \cdot e^{cos(t)} + c$
   D. $(\cos(t) - \sin^2(t)) \cdot e^{\cos(t)} + c$
   E. Cannot evaluate

19. A function $h(x)$ is continuous for all x-values. Given that $-x^2 + 1 \leq h(x) \leq cos\left(\frac{\pi}{3}x\right)$, evaluate $\lim_{x \to 1} h(x)$.
   A. 0
   B. 1/2
   C. 1
   D. 2
   E. Not enough information given

20. Evaluate the limit

$$\lim_{x\to\infty} \frac{e^{x/10}}{x^3} =$$

A. 0
B. $e/3$
C. $1/2$
D. $1/10$
E. ∞

21. Given that $y = \sin(x)$. Find the 23rd derivative of y; that is, find $y^{(23)}(x)$.
A. $y^{(23)}(x) = -\cos(x)$
B. $y^{(23)}(x) = -\sin(x)$
C. $y^{(23)}(x) = \cos(x)$
D. $y^{(23)}(x) = \sin(x)$
E. None of the above.

22. Calculate the total area bounded by the curves $y = sin(x)$ and $y = cos(x)$ between $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.
A. 0
B. $\sqrt{2}$
C. 2
D. $2\sqrt{2}$
E. None of the above.

23. Evaluate the following limit:

$$\lim_{h\to 0} \frac{|h|}{h} =$$

A. -1
B. 1
C. $-\infty$
D. ∞
E. The limit does not exist

24. The point $(3,3)$ lies on the curve $x^3 + y^3 = 6xy$. Find the slope of the tangent line at this point.
    A. $\frac{dy}{dx} = 0$
    B. $\frac{dy}{dx} = -1$
    C. $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{3}$
    D. $\frac{\mathrm{dy}}{\mathrm{dx}} = 2$
    E. Not enough information given.

25. Rolle's Theorem does not apply to $f(x) = |x - 5|$ on the interval $[3,6]$ because
    A. $f(x)$ is not continuous on $[3,6]$
    B. $f(x)$ is not continuous on $(3,6)$
    C. $f(3) \neq f(6)$
    D. $f'(3) > f'(6)$
    E. None of these

Tie Breaker #1

Name: ____________________

School: ___________________

Define the function $f(x) = x^a \cdot (x-1)^b$, for two positive constants a and b.

What is the x-value where $f(x)$ reaches an extrema on the interval $0 < x < 1$?

Tie Breaker #2

Name: ___________________

School: __________________

Suppose you deposit a principle of $10,000 into a savings account with interest rate of 3%. Your bank calculates your balance monthly ($m = 12$) based off the deposited principle as well as previously compounded interest. The amount at time t is found by the formula:

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

A. Determine a function to give the instantaneous rate of change for $A(t)$.
B. Evaluate your function found in part A to determine the instantaneous rate of change when $t = 2$.
C. Write a short explanation to give an interpretation to what your value in part B means.

Tie Breaker #3

Name: ____________________

School: ____________________

Find the exact area bounded by the following curves. Show all work.

$$y = 1$$
$$y = -1$$
$$x = e^y$$
$$x = y^2 - 2$$

Solutions to 2020 Regional Calculus Exam

1	D	11	C	21	A
2	C	12	A	22	D
3	C	13	B	23	E
4	C	14	C	24	B
5	A	15	B	25	C
6	C	16	D		
7	A	17	D		
8	C	18	A		
9	D	19	E		
10	D	20	E		

Tie Breaker 1 Answer

Define the function $f(x) = x^a \cdot (x-1)^b$, for two positive constants a and b.

What is the x-value where $f(x)$ reaches its extrema on the interval $0 < x < 1$?

First find critical points. We will set the derivative equal to zero and solve for x.

$$f'(x) = x^a \cdot b \cdot (x-1)^{b-1} + a \cdot x^{a-1} \cdot (x-1)^b$$

Factor by grouping.

$$f'(x) = b \cdot x \cdot x^{a-1} \cdot (x-1)^{b-1} + a \cdot x^{a-1} \cdot (x-1) \cdot (x-1)^{b-1}$$
$$f'(x) = \big(b \cdot x + a \cdot (x-1)\big) \cdot x^{a-1} \cdot (x-1)^{b-1}$$
$$f'(x) = \big((a+b) \cdot x - a\big) \cdot x^{a-1} \cdot (x-1)^{b-1}$$

Set each factor equal to zero and solve.

$$(a+b) \cdot x - a = 0 \quad \rightarrow \quad x = \frac{a}{a+b}$$

$x^{a-1} = 0 \quad \rightarrow \quad x = 0$ which is not in the interval so we disregard it.

$(x-1)^{b-1} = 0 \quad \rightarrow \quad x = 1$ which is not in the interval so we disregard it.

Thus, the value for x where $f(x)$ reaches its extrema is $x = \frac{a}{a+b}$.

We know that $x = \frac{a}{a+b}$ lies in the given interval because $0 < \frac{a}{a+b} < \frac{a}{a} = 1$ when a and b are positive values.

Tie Breaker 2 Answer

Suppose you deposit a principle of $10,000 into a savings account with interest rate of 3%. Your bank calculates your balance monthly ($m = 12$) based off the deposited principle as well as previously compounded interest. The amount at time t is found by the formula:

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

A. *Determine a function to give the instantaneous rate of change for $A(t)$.*
B. *Evaluate your function found in part A to determine the instantaneous rate of change when $t = 2$.*
C. *Write a short explanation to give an interpretation to what your value in part B means.*

A) First define our formula using the values given.

$$A(t) = 10000\left(1 + \frac{0.03}{12}\right)^{12t} = 10000(1.0025)^{12t}$$

Now find the derivative of $A(t)$, which is the instantaneous rate of change formula.

$$A'(t) = 10000(1.0025)^{12t} \cdot \ln(1.0025) \cdot 12$$
$$A'(t) = 120000 \ln(1.0025) \cdot (1.0025)^{12t}$$

B) At $t = 2$, we evaluate the derivative found in part A.

$$A'(2) = 120000 \ln(1.0025) \cdot (1.0025)^{12 \cdot 2}$$
$$A'(2) = 318.13$$

C) Answers will vary for this part, but generally they take the form of

"At $t = 2$ years, the balance in the account will increase by $318.13 per year."

Tie Breaker 3 Answer

Find the exact area bounded by the following curves. Show all work.

$$y = 1$$
$$y = -1$$
$$x = e^y$$
$$x = y^2 - 2$$

On the interval $-1 < y < 1,\ e^y > y^2 - 2$.

$$Area = \int_{-1}^{1} e^y - (y^2 - 2)\,dy = \int_{-1}^{1} e^y - y^2 + 2\,dy = \left(e^y - \frac{y^3}{3} + 2y\right|_{-1}^{1}$$
$$= \left(e^1 - \frac{1}{3} + 2\right) - \left(e^{-1} - \frac{-1}{3} - 2\right) = e - \frac{1}{3} + 2 - \frac{1}{e} - \frac{1}{3} + 2$$
$$= e - \frac{1}{e} + \frac{10}{3}\,units^2 \text{ exact answer}$$
$$\approx 5.684\ units^2 \text{ approximate answer}$$