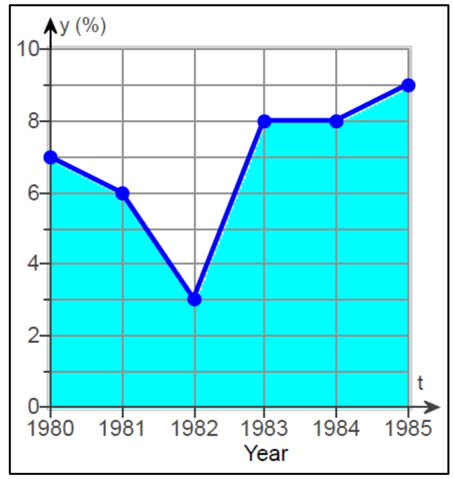
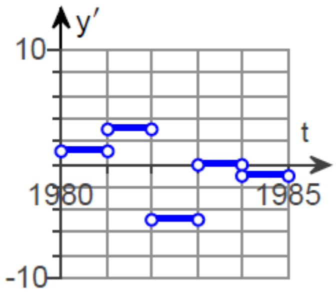
Arkansas Council of Teachers of Mathematics **Name ________________________**

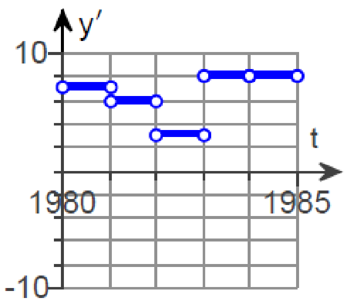
<u>2022 Calculus Regional Competition</u>

Work the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then attempt the tie-breaker questions at the end starting with tie breaker #1, then #2, and then #3. Turn in your answer sheet, your tie-breaker pages, and your scratch work when you are finished. Figures are not necessarily drawn to scale. Angles are given in radians unless otherwise stated.

1. The graph in the figure on the right shows the average annual percentage change $y = f(t)$ in a particular country's gross domestic product (GDP) for the years 1980-1985. Choose the correct graph of $\frac{dy}{dt}$.



A.


B.


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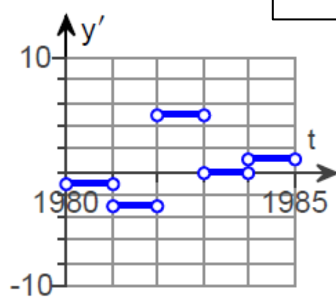
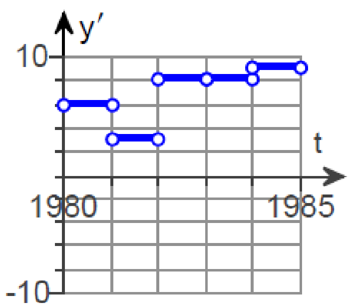
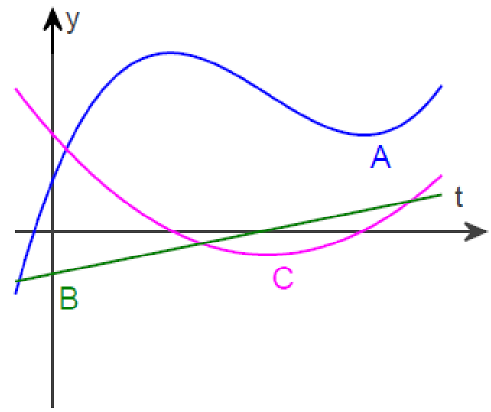


D.


E. None of the other options.

2. Approximate the area bounded by $f(x) = \frac{1}{x}$, between $x = 4$, $x = 5$, and the x-axis using a Riemann Sum. Use two rectangles with right-hand endpoints and equal sub-interval widths.
   a. $\frac{17}{72}$
   b. $\frac{19}{72}$
   c. $\frac{17}{90}$
   d. $\frac{19}{90}$
   e. None of the above

3. The graphs in the beside figure show the position s, velocity $v = \frac{ds}{dt}$, and acceleration $a = \frac{d^2s}{dt^2}$ of a body moving along a coordinate line as functions of time t. Which graph is which?



   a. The graph labeled A is the graph of the position s, the graph labeled C is the graph of the velocity v, and the graph labeled B is the graph of the acceleration a.

   b. The graph labeled C is the graph of the position s, the graph labeled A is the graph of the velocity v, and the graph labeled B is the graph of the acceleration a.

   c. The graph labeled A is the graph of the position s, the graph labeled B is the graph of the velocity v, and the graph labeled C is the graph of the acceleration a.

   d. The graph labeled B is the graph of the position s, the graph labeled C is the graph of the velocity v, and the graph labeled A is the graph of the acceleration a.

   e. The graph labeled C is the graph of the position s, the graph labeled B is the graph of the velocity v, and the graph labeled A is the graph of the acceleration a.

4. On what interval(s) is the function $f(x) = \frac{x-x^2}{1+3x^2}$ increasing?
   a. $\left(-\frac{\sqrt{3}}{3}i, \frac{\sqrt{3}}{3}i\right)$
   b. $\left(-1, \frac{1}{3}\right)$
   c. $\left(-\frac{1}{2}, \frac{1}{6}\right)$
   d. $f(x)$ never increases.
   e. None of the above.

5. Complete this sentence with the best option. If f is a polynomial function of degree n, then f has…
   a. at least one critical number.
   b. n critical numbers.
   c. $(n-1)$ critical numbers.
   d. no more than $(n-1)$ critical numbers.
   e. no more than $(n-2)$ critical numbers.

6. What is the slope of the line normal to the graph of $y = x^3 - 3x^2 + 6x + 2022$ at its point of inflection?
   a. 1
   b. -1
   c. $-\frac{1}{3}$
   d. 3
   e. Cannot be determined.

7. Evaluate the following limit.

$$\lim_{x\to\infty} \frac{\ln\left(\sqrt[7]{x}\right)}{\log\left(\sqrt[3]{x}\right)} =$$

   a. 1
   b. $\frac{7}{3}$
   c. $\frac{3\ln(10)}{7}$
   d. $\frac{e}{10}$
   e. Limit does not exist, *or* Limit cannot be determined

8. Iodine-131 is a radioactive element used in medicine. It has a half-life of about 8 days. Suppose a patient is given a 100 millicurie (written 100 mCi) dose of this medication. What is the rate of change (in $\frac{mCi}{day}$) at $t = 12$ days?
   a. $-3.0\,\frac{mCi}{day}$
   b. $-1.1\,\frac{mCi}{day}$
   c. $-0.1\,\frac{mCi}{day}$
   d. $25\,\frac{mCi}{day}$
   e. None of the above.

9. Evaluate the limit

$$\lim_{h\to 8} \frac{\frac{1}{h} - \frac{1}{8}}{h - 8} =$$

   a. $-\frac{1}{8}$
   b. $\frac{1}{2}$
   c. $-\frac{1}{16}$
   d. $\frac{1}{4}$
   e. None of the above.

10. Evaluate. Assume all variables are non-negative:

$$\frac{d}{dx}\left[\int_0^x \sqrt{10t + 9}\, dt\right] =$$

   a. $\frac{5}{\sqrt{10x+9}}$
   b. $\sqrt{10x + 9}$
   c. $\frac{1}{15}(10x + 9)^{3/2}$
   d. $\sqrt{10x + 9} - \sqrt{9}$
   e. Unable to determine

11. For an unknown function $f(x)$ continuous at $x = 0$, $\sqrt{10 - 3x^2} \leq f(x) \leq \sqrt{10 - x^2}$ on the interval $-1 \leq x \leq 1$. Use this fact to evaluate $\lim_{x \to 0} f(x)$.
    a. $-\sqrt{7}$
    b. $\sqrt{10}$
    c. 7
    d. 10
    e. Does not exist *or* Cannot be determined

12. Consider the following piecewise function. Is this function differentiable at $x = 1$? Choose the best explanation why or why not.
$$f(x) = \begin{cases} x+1 & x \leq 1 \\ -3x+5 & x > 1 \end{cases}$$
    a. Yes, $f(x)$ is differentiable because we can find the derivative of the left and the right "branches" of $f(x)$.
    b. Yes, $f(x)$ is differentiable because $f(x)$ is continuous at $x = 1$.
    c. No, $f(x)$ is not differentiable because $f(x)$ has a corner at $x = 1$.
    d. No, $f(x)$ is not differentiable because $f(x)$ is a piecewise function.
    e. None of the above.

13. Define $f(x) = x^3 + 6x^2 + 14$. Define values a & b as the two critical numbers for $f(x)$, with $a \leq b$. Evaluate
$$\int_a^b f(x)\,dx =$$
    a. -6.4
    b. 194.5
    c. 248
    d. 368
    e. None of the above.

14. For what values of a and b is the following piecewise function continuous for all x-values?
$$f(x) = \begin{cases} -2 & x \leq -1 \\ ax+b & -1 < x \leq 3 \\ 12 & 3 < x \end{cases}$$
    a. $a = \frac{12}{13}$ & $b = \frac{16}{23}$
    b. $a = \frac{12}{13}$ & $b = \frac{16}{13}$
    c. $a = 12$ & $b = 6$
    d. $a = \frac{7}{2}$ & $b = \frac{3}{2}$
    e. No value(s) exist *or* Cannot be determined

15. You're standing on a tree house that is $5m$ above the ground. You throw an object upward with an initial velocity of $10\frac{m}{s}$. What is the velocity of the object at the instant it hits the ground? Approximate g with $10\frac{m}{s^2}$.
    a. $0\frac{m}{s}$
    b. $-10\frac{m}{s}$
    c. $-14.1\frac{m}{s}$
    d. $-70\frac{m}{s}$
    e. None of the above.

16. Build off your work in the previous question. What is the velocity of the object when it reaches its maximum height?
    a. $0\frac{m}{s^2}$
    b. $-10\frac{m}{s^2}$
    c. $-14.1\frac{m}{s^2}$
    d. $-70\frac{m}{s^2}$
    e. None of the above.

17. Find the slope of the tangent line of the following curve at the given point.

$$y^4 + x^3 = y^2 + 10x \quad point = (0, 1)$$

    a. $m_{tan} = -5$
    b. $m_{tan} = 5$
    c. $m_{tan} = \frac{5}{2}$
    d. $m_{tan} = -\frac{5}{2}$
    e. None of the above

18. Evaluate $\int \sec(x)\tan(x)\,dx$:
    a. $-\cot(x) + c$
    b. $\tan(x) + c$
    c. $\sec(x) + c$
    d. $-\csc(x) + c$
    e. None of the above

19. Find the point(s) c on the interval $[-6, 3]$ at which function $h(x) = 4 - x^2$ equals its average height on the interval.
    a. $c = \pm 2$
    b. $c = \pm 3$
    c. $c = \pm\sqrt{7}$
    d. $c = \pm\sqrt{10}$
    e. None of the above.

20. Find the following anti-derivative.

$$\int \frac{\sin(t)}{(3 + \cos(t))^5}\,dt$$

    a. $\frac{1}{6}(3 + \cos(t))^{-6} + C$
    b. $4(3 + \cos(t))^{-4} + C$
    c. $(3 + \cos(t))^{-4} + C$
    d. $\frac{1}{4}(3 + \cos(t))^{-4} + C$
    e. None of the above

21. Find the volume of revolution for the solid generated by revolving the region bounded between these two equations about the x-axis.

$$y = \frac{8}{x} \quad \& \quad y = -x + 9$$

 a. $\frac{3745}{3}$
 b. $\frac{3745\pi}{3}$
 c. $\frac{343}{3}$
 d. $\frac{343\pi}{3}$
 e. None of the above

22. The Mean Value Theorem applies to the function $f(x) = x^{5/2}$ on the interval $[0, 1]$. Find all possible values of c that satisfy the MVT.
 a. $c = \left(\frac{2}{5}\right)^{2/3}$
 b. $c = \left(\frac{2}{5}\right)^{3/2}$
 c. $c = \left(\frac{5}{2}\right)^{2/3}$
 d. $c = \left(\frac{5}{2}\right)^{3/2}$
 e. No value exists

23. Calculate $\frac{d}{dx}[\sin(\sin(x))]$.
 a. $\cos(cos(x))$
 b. $\cos(x) \cdot \cos(sin(x))$
 c. $\sin(x) \cdot \sin(cos(x))$
 d. $\cos^2(x)$
 e. None of the above

24. Consider the function $f(x) = x^2 + \frac{b}{x}$. What value of b makes this $f(x)$ have a local minimum at $x = -2$?
 a. $b = -6$
 b. $b = 6$
 c. $b = -16$
 d. $b = 16$
 e. No value b can be found.

25. Find $\frac{dy}{dx}$: $e^{7x} = \sin(x + 7y)$.
 a. $\frac{dy}{dx} = \frac{e^{7x}}{\cos(x+7y)} - \frac{1}{7}$
 b. $\frac{dy}{dx} = \frac{2e^{7x}}{3\cos(x+3y)} - \frac{1}{3}$
 c. $\frac{dy}{dx} = \frac{-2e^{7x}}{3\sin(x+3y)} + \frac{1}{3}$
 d. $\frac{dy}{dx} = \frac{-e^{7x}}{\sin(x+7y)} + \frac{1}{7}$
 e. Cannot be found.

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TIE BREAKER #1

Name: ________________________

School: ________________________

Find the derivative of the following function, using one single formula.

$$f(x) = |x|^3$$

TIE BREAKER #2

Name: ________________________

School: ________________________

Solve the initial value problem to find y(x). Use exact values. Show all work. Assume all values for x are positive.

$$\frac{dy}{dx} = \frac{1}{x^3} + x \quad \& \quad y(3) = 1$$

TIE BREAKER #3

Name: ______________________

School: ______________________

Find a cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ such that the graph of p has a local maximum at point $(-3, 10)$ and a point of inflection when $x = -\frac{5}{3}$.

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ANSWER KEY

# 1	C	# 6	C	# 11	B	# 16	A	# 21	D
# 2	D	# 7	C	# 12	C	# 17	B	# 22	A
# 3	A	# 8	A	# 13	E	# 18	C	# 23	B
# 4	B	# 9	E	# 14	D	# 19	B	# 24	C
# 5	D	# 10	B	# 15	C	# 20	D	# 25	A

Tie Breaker 1

Since $|x| = \sqrt{x^2}$, you can revise the given formula to $f(x) = |x|^3 = \left(\sqrt{x^2}\right)^3 = (x^2)^{3/2}$. We can apply the Chain Rule to this formula.

$$f'(x) = \frac{3}{2}(x^2)^{3/2-1} \cdot (2x) = \frac{3}{2}(x^2)^{1/2} \cdot 2x = 3x\sqrt{x^2}$$
$$= 3x|x|$$

Tie Breaker 2

$$\frac{dy}{dx} = \frac{1}{x^3} + x \quad \& \quad y(3) = 1$$

This is a separable differential equation. We integrate:

$$\int dy = \int \frac{1}{x^3} + x\, dx$$
$$\int dy = \int x^{-3} + x\, dx$$
$$y = \frac{x^{-2}}{-2} + \frac{x^2}{2} + c = -\frac{1}{2x^2} + \frac{x^2}{2} + c$$

Evaluate this at coordinate $(3, 1)$ and solve for c.

$$1 = -\frac{1}{2 \cdot 3^2} + \frac{3^2}{2} + c$$
$$c = -\frac{31}{9} = -3.\overline{4}$$

Final answer:

$$\boldsymbol{y = -\frac{1}{2x^2} + x^2 - \frac{31}{9}}$$

Tie Breaker 3

Define $f(x) = x^3 + ax^2 + bx + c$. Consider the inflection point first. We know $\frac{d^2}{dx^2}[f(x)] = 0$ at $x = -\frac{5}{3}$.

$$\frac{d^2}{dx^2}[f(x)] = 6x + 2a = 0$$

Solve for a at point $x = -\frac{5}{3}$: $a = 5$.

Redefine $f(x) = x^3 + 5x^2 + bx + c$. Now consider the critical point at $x = -3$.

$$\frac{d}{dx}[f(x)] = 3x^2 + 10x + b = 0$$

Solve for b at point $x = -3$: $b = 3$.

Redefine $f(x) = x^3 + 5x^2 + 3x + c$. We now consider the coordinate $(-3, 10)$.

$$f(-3) = 10$$
$$(-3)^3 + 5(-3)^2 + 3(-3) + c = 10$$
$$\boldsymbol{c = 1}$$

Final answer: $\boldsymbol{f(x) = x^3 + 5x^2 + 3x + 1}$.