ACTM State Calculus Competition
2017

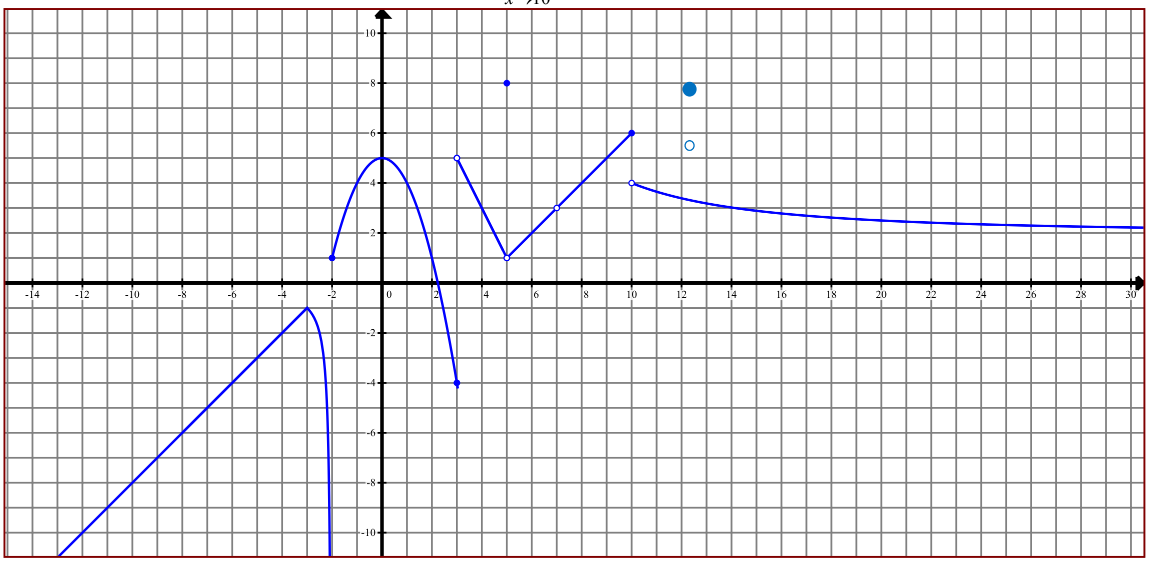
Begin by removing the three tie breaker sheets at the end of the exam and writing your name on all three pages. Work the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Make sure you attempt the tie-breaker questions at the end of the test starting with tie breaker 1, then 2, and then 3 if you have time. Turn in your answer sheet and the tie breaker pages when you are finished. You may keep the pages with the multiple-choice questions.

1. $\lim_{x\to 2}\frac{(3x+4)(x-2)}{(x-2)}=10$. By the definition of a limit, for every $\varepsilon>0$ there exists a positive real number δ such that $\left|\frac{(3x+4)(x-2)}{(x-2)}-10\right|<\varepsilon$ if $0<|x-2|<\delta$. The largest valid value of δ for a particular value of ε which makes these inequalities true is
   A. $\frac{\varepsilon}{4}$
   B. 4ε
   C. 3ε
   D. $\frac{\varepsilon}{3}$
   E. $\frac{4\varepsilon}{3}$

2. $\lim_{x\to\infty}\left(\frac{x+3}{x-4}\right)^x=$
   A. 0
   B. 1
   C. 7
   D. e^7
   E. ∞

3. Which of the following indicates the presence of a vertical asymptote for the graph of $y=f(x)$?
   A. $\lim_{x\to 3}f(x)=\infty$
   B. $\lim_{x\to\infty}f(x)=3$
   C. $\lim_{x\to\infty}f(x)=\infty$
   D. $\lim_{x\to 4}f(x)=3$
   E. Each of the other answers is incorrect.

4. Given the following graph of $f(x)$, what is $\lim_{x \to 10^-} f(x)$?



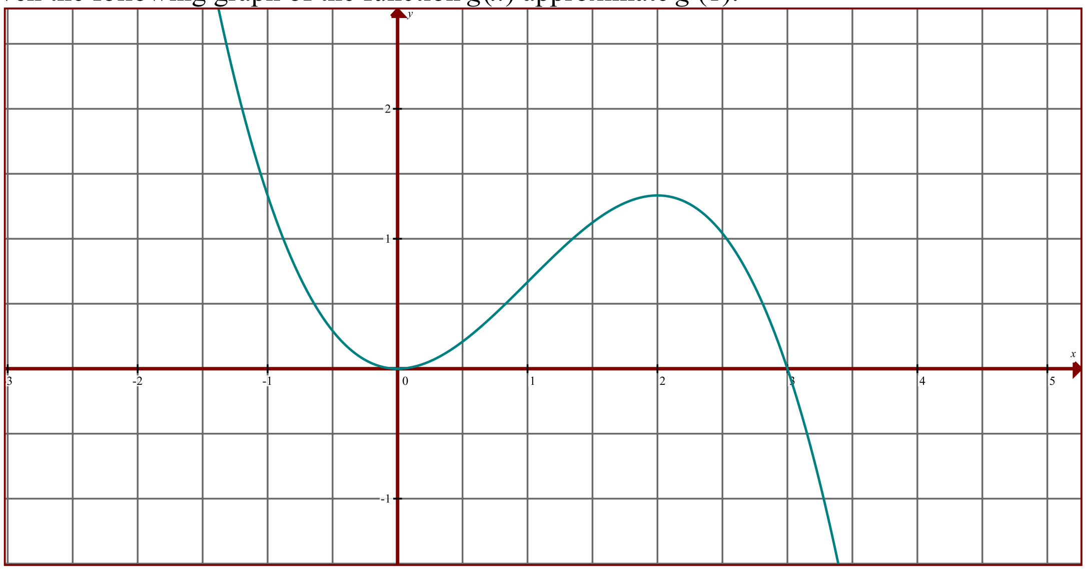
A. -8
B. 4
C. 6
D. 8
E. Does not exist

5. Which of the following statements is false?
   A. If a function is differentiable for a particular x-value then it must be continuous at that x-value.
   B. If a function has different limits from the left and from the right at a particular x-value, then the function is discontinuous at that x-value.
   C. If a function is continuous for a particular x-value then it must be differentiable at that x-value.
   D. If a function is undefined for a particular x-value then its derivative is undefined at that x-value.
   E. Polynomial functions are differentiable for all real values of the input variable.

6. If $p(2) = 3$, $p'(2) = 0$ and $p''(2) = -4$ which of the following must be true about the graph of $p(x)$?
   A. The graph has a local maximum at (2, 3).
   B. The graph has a local minimum at (2, 3).
   C. The graph has an inflection point at (2, 3).
   D. There is a hole in the graph at (2, 3).
   E. Each of the other answers is incorrect.

7. If $s(2) = 3$, $s'(2) = 0$ and $s''(2) = 0$ which of the following must be true about the graph of $s(x)$?
   A. The graph has a local maximum at (2, 3).
   B. The graph has a local minimum at (2, 3).
   C. The graph has an inflection point at (2, 3).
   D. There is a hole in the graph at (2, 3).
   E. Each of the other answers is incorrect.

8. Given the following graph of the function $g(x)$ approximate $g'(1)$.



A. -2
B. $-\frac{1}{2}$
C. 0
D. $\frac{1}{2}$
E. 2

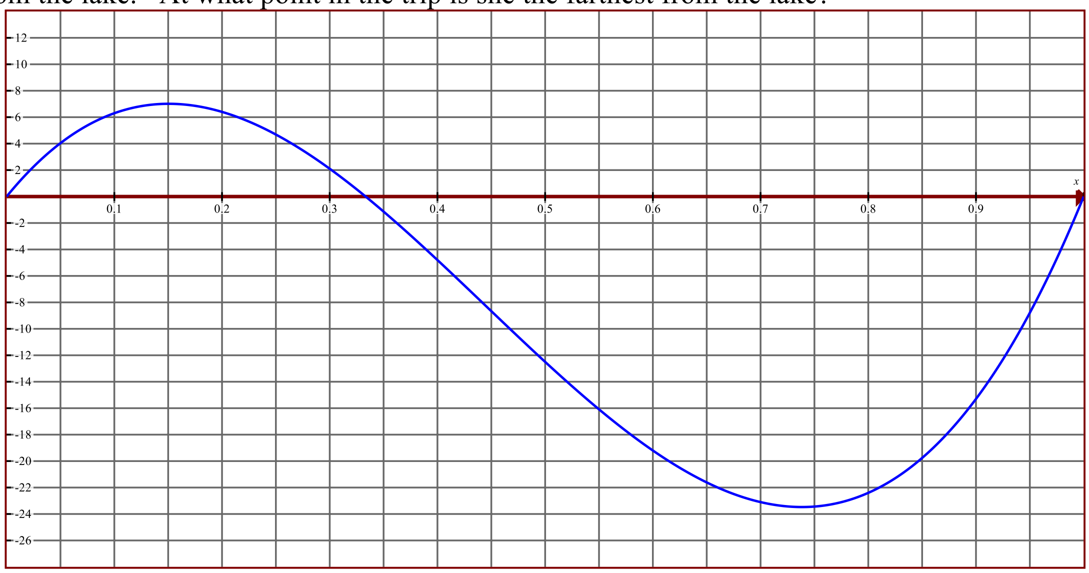
9. Let $f(x) = x^{2x}$ then $f'(2) \approx$
A. 4.00
B. 11.09
C. 16.00
D. 32.00
E. 54.18

10. A baseball team plays in a stadium that hold 51,000 spectators. With ticket prices at \$10 the average attendance had been 38,000. When ticket prices were lowered to \$8 the average attendance rose to 42,000. Let us assume the demand function is linear. What should they set the ticket price to in order to maximize the revenue from ticket sales?
A. \$9.20/ticket
B. \$10.00/ticket
C. \$14.50/ticket
D. \$18.60/ticket
E. Each of the other answers is incorrect.

11. A major league baseball diamond is a square 90 feet along each side. A player hits a ball down the third base line at a speed of 125 feet per second. He immediately runs toward first base at a rate of 25 feet per second. To the nearest whole number, how fast is the distance between the ball and the runner increasing at exactly 2 seconds after the ball leaves the bat?
A. 100 ft/sec
B. 150 ft/sec
C. 217 ft/sec
D. 255 ft/sec
E. Each of the other answers is incorrect.

12. The population of a certain region is currently 35,455 and is increasing at a constant percentage rate of 4% per year. What is the average population of the region over the next ten years?
    A. 43,413
    B. 43,594
    C. 43,969
    D. 44,174
    E. Each of the other answers is incorrect.

13. A bicyclist is pedaling along a straight road with velocity, v, (in miles per hour) given in the graph below as a function of the time since starting in hours. Suppose the cyclist starts 5 miles from the lake, and that positive velocities take her *toward* the lake and negative velocities take her *away* from the lake. At what point in the trip is she the farthest from the lake?



    A. 0 hours
    B. 0.15 hours
    C. 0.33 hours
    D. 0.75 hours
    E. 1 hour

14. $\frac{d^n}{dx^n}\left(\ln(x)\right) =$
    A. $\dfrac{(-1)^{n-1}\left[(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1\right]}{x^n}$
    B. $\dfrac{(-1)^{n+1}\left[n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1\right]}{x^{n+1}}$
    C. $-\dfrac{1}{x^n}$
    D. $\dfrac{1-n}{x^n}$
    E. $-\dfrac{\left[n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1\right]}{x^{n+1}}$

15. *The rate of change of the slope of a function* is a description of the
   A. limit.
   B. first derivative.
   C. second derivative.
   D. definite integral.
   E. indefinite integral.

16. Which of the following is an antiderivative of $f(x)=e^{-x^2}$?
   A. $f(x)=e^{-x^2}$
   B. $h(x)=-2xe^{-x^2}$
   C. $k(x)=-\frac{1}{2x}e^{-x^2}$
   D. $m(x)=\int_2^x e^{-t^2}\,dt$
   E. Each of the other answers is incorrect.

17. Function f is strictly increasing and continuous on [0, 4], and has continuous first and second derivatives on [0,4]. $f''(x)<0$ when $0\le x<2$ and $f''(x)>0$ when $2<x\le 4$. Therefore,
   A. The function f attains a relative maximum value at $x=2$.
   B. The function f attains a relative minimum value at $x=2$.
   C. The rate of change of function f attains a relative maximum value at $x=2$.
   D. The rate of change of function f attains a relative minimum value at $x=2$.
   E. None of the other answers is correct.

18. *The cumulative effect of a function acting over an interval* is a description of the
   A. limit.
   B. first derivative.
   C. second derivative.
   D. definite integral.
   E. indefinite integral.

19. $\int_1^3 \left(3x^2+1\right)dx =$

A. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(3\left(1+\frac{2}{n}i\right)^2+1\right)\frac{2}{n}$

B. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(1+\frac{2}{n}i\right)^3+\left(1+\frac{2}{n}i\right)\right)\frac{2}{n}$

C. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\left(\frac{3}{n}i\right)^3+\frac{3}{n}i\right)+\left(\left(\frac{1}{n}i\right)^3+\frac{1}{n}i\right)\right)\frac{2}{n}$

D. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(3\left(1+\frac{2}{n}i\right)^2+1\right)\left(\frac{2}{n}i\right)$

E. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(3\left(\frac{3}{n}i\right)^2+1\right)\left(\frac{2}{n}i\right)$

20. The velocity of a scooter was recorded at 15 second intervals over the first 2 minutes of a trip:

Time, *t* min:sec	0:00	0:15	0:30	0:45	1:00	1:15	1:30	1:45	2:00
Velocity, km/hr	0	12	23	29	32	34	28	25	31

Using **30** second intervals, approximate the distance traveled using a *midpoint* approximation.

A. $\frac{5}{12}$ km

B. $\frac{5}{6}$ km

C. $\frac{197}{240}$ km

D. $\frac{107}{120}$ km

E. 3000 km

21. *Bled Dead Zombie XV* is the latest rage in video games. Development costs of the game (programmers, artists, equipment, workspace, etc.) totaled \$3,150,256. Once the game is put into production, the additional cost of printing, packaging, and shipping a copy averages \$8 per copy. The game becomes a world-wide phenomenon, selling hundreds of millions of copies. As the number of copies produced and sold continues to increase, the average cost per copy $\overline{C}$ will

A. Increase at a constant rate.

B. Increase at a decreasing rate, approaching a horizontal asymptote $\overline{C}=31$

C. Decrease, approaching a horizontal asymptote $\overline{C}=8$.

D. Decrease, approaching a horizontal asymptote $\overline{C}=0$.

E. None of the other answers is correct.

22. $\lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x^3}} =$

A. -1
B. 0
C. $\frac{3}{2}$
D. 2
E. ∞

23. $f(x)=\sqrt{x}$. Find the equation of the line tangent to the *derivative* $y=f'(x)$ at $x=4$.

A. $y=2+\frac{1}{4}(x-4)$
B. $y=2-\frac{1}{32}(x-4)$
C. $y=\frac{1}{4}-\frac{1}{32}(x-4)$
D. $y=\frac{1}{4}+\frac{1}{4}(x-4)$
E. $y=\frac{1}{4}+\frac{1}{4}(x-2)$

24. The base of a solid is the region above $y=x^2$ and below $y=4$. Cross sections of the solid cut perpendicular to the *x*-axis are semicircles. The volume of the solid is

A. $\frac{1}{2}\pi\int_{-2}^{2}\left(4-x^2\right)^2 dx$
B. $\frac{1}{2}\pi\int_{-2}^{2}\left(16-x^4\right)dx$
C. $\frac{1}{2}\pi\int_{-2}^{2}\left(\frac{4-x^2}{2}\right)^2 dx$
D. $\frac{1}{2}\pi\int_{0}^{4} y\,dy$
E. $\frac{1}{2}\pi\int_{0}^{4}\left(\frac{4+y}{2}\right)^2 dy$

25. $\int\left(2e^{2x}\sin(3x)+3e^{2x}\cos(3x)\right)dx =$

A. $-2e^{2x}\cos(3x)+3e^{2x}\sin(3x)+C$
B. $-\frac{1}{3}e^{2x}\cos(3x)+\frac{1}{2}e^{2x}\sin(3x)+C$
C. $12e^{2x}\cos(3x)-5e^{2x}\sin(3x)+C$
D. $-\frac{1}{6}e^{2x}\cos(3x)+C$
E. $e^{2x}\sin(3x)+C$

Name ________________________________

Tie Breaker 1:

In statistics, it is important to know that the total area under a curve equals a specific value. Find the value of the base *a* in the following integral that results in a limit of 1:

$$\lim_{b\to\infty}\left(\int_0^b a^x\,dx\right)=1$$

Name ________________________________

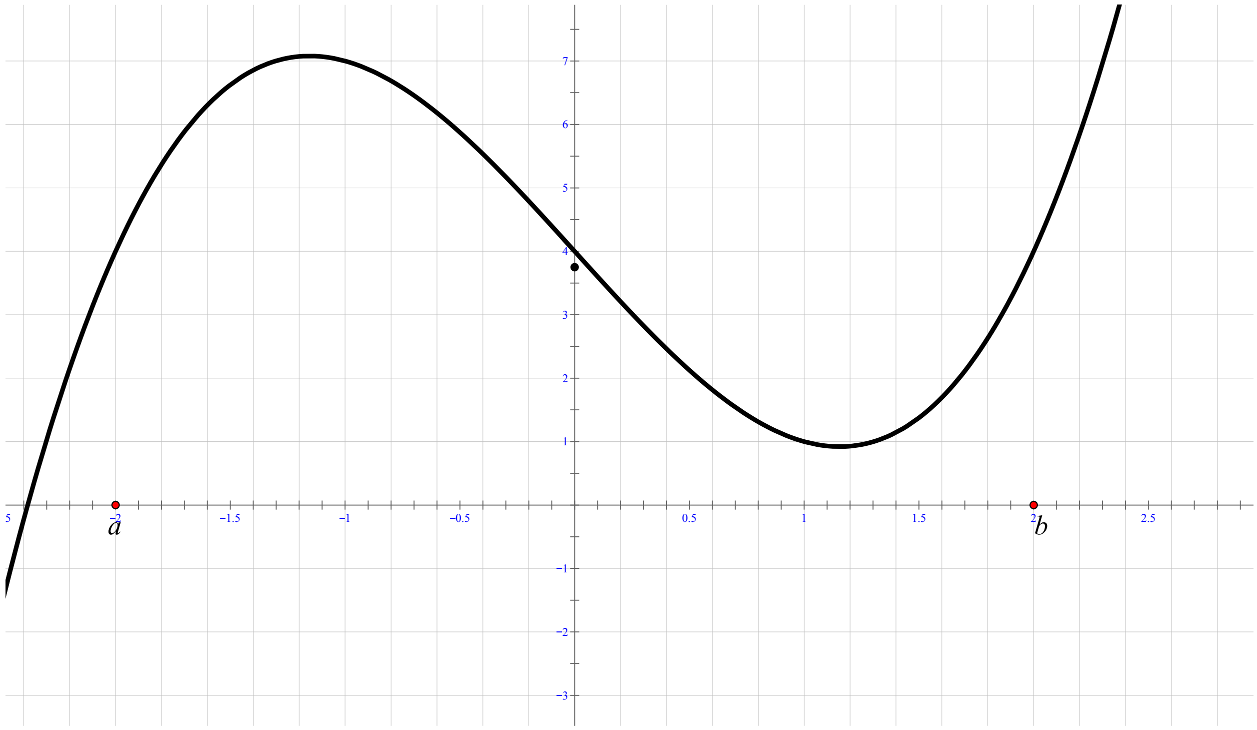
Tie Breaker 2:

Suppose that we have a function which is continuous and has positive values on an interval [*a, b*]. We want to approximate $\int_a^b f(x)\,dx$ with the Midpoint Rule. Note that one way to visualize this approximation is to approximate the desired area with the sum of areas of rectangles.

- Illustrate these rectangles on the graph of the function on the following page using $n = 4$ subdivisions and $[a, b] = [-2, 2]$.

Next, we see that we can also visualize this same approximated area given by the Midpoint Rule as the sum of the areas of right trapezoids with the tops of the trapezoids being line segments which are tangent to the curve at the horizontal middle of each interval.

- Add these right trapezoids to your illustration.
- Prove that these two areas are equivalent.
- Use this information to answer the following questions, circling one correct answer per question below
    - If the function is linear on each of the subintervals then the midpoint rules is an ________ estimate of the integral.
        A. Exact
        B. Under
        C. Over
    - If the function is concave down on each of the subintervals then the midpoint rules is an ________ estimate of the integral.
        A. Exact
        B. Under
        C. Over
    - If the function is concave up on each of the subintervals then the midpoint rules is an ________ estimate of the integral.
        A. Exact
        B. Under
        C. Over

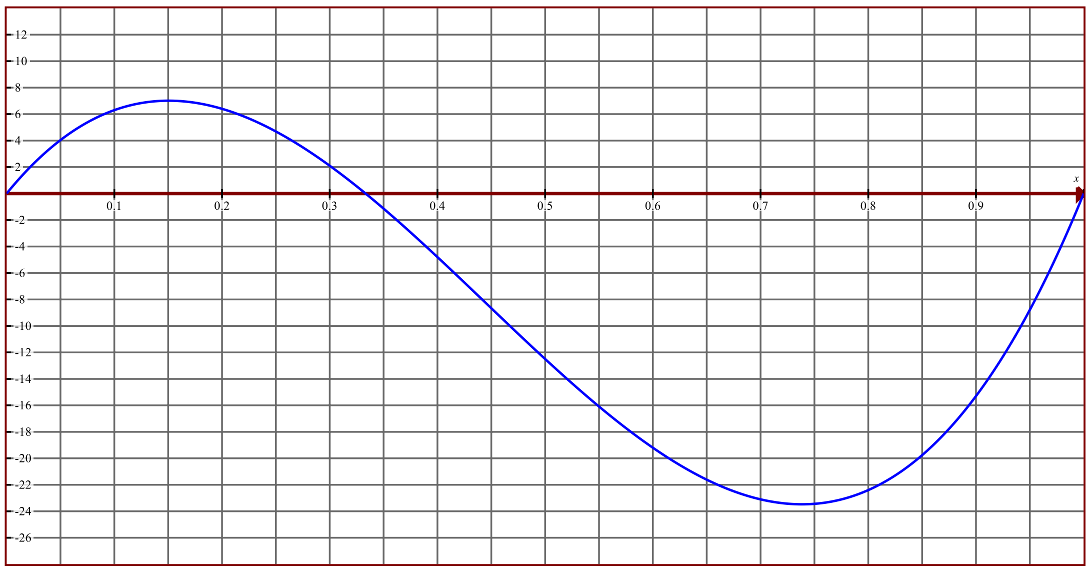


7
6
5
4
3
2
1
-1
-2
-3
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
a
b

Name ________________________________

Tie Breaker 3:

A bicyclist is pedaling along a straight road with velocity, v, (in miles per hour) given in the graph below as a function of the time since starting in hours. Suppose the cyclist starts 5 miles from the lake, and that positive velocities take her *toward* the lake and negative velocities take her *away* from the lake. At the point she is farthest from the lake how far away is she?



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Key

1.	D
2.	D
3.	A
4.	C
5.	C
6.	A
7.	E
8.	D
9.	E
10.	C
11.	E
12.	A
13.	E
14.	A
15.	C
16.	D
17.	D
18.	D
19.	A
20.	B
21.	C
22.	D
23.	C
24.	C
25.	E

Tie Breaker 1:

In statistics, it is important to know that the total area under a curve equals a specific value. Find the positive real value of the base a in the following integral that results in a limit of 1:

$$\lim_{b\to\infty}\left(\int_0^b a^x dx\right)=1$$

Solution:

If a = 1 then $\lim_{b\to\infty}\left(\int_0^b a^x dx\right)=\lim_{b\to\infty}\left(\int_0^b 1dx\right)=\lim_{b\to\infty}x\Big|_0^b=\lim_{b\to\infty}b=\infty$.

If $a\neq 1$ then $\lim_{b\to\infty}\left(\int_0^b a^x dx\right)=\lim_{b\to\infty}\left(\frac{a^x}{\ln(a)}\Bigg|_0^b\right)=\lim_{b\to\infty}\left(\frac{a^b-1}{\ln(a)}\right)$

If $a>1$, the limit does not exist.

Thus, any solution must satisfy $0<a<1$. In this case, $\lim_{b\to\infty}\left(a^b\right)=0$, making

$\lim_{b\to\infty}\left(\int_0^b a^x dx\right)=\lim_{b\to\infty}\left(\frac{a^b-1}{\ln(a)}\right)=\frac{-1}{\ln(a)}$

Our equation $\lim_{b\to\infty}\left(\int_0^b a^x dx\right)=1$ becomes $\frac{-1}{\ln(a)}=1 \quad\Rightarrow\quad \ln(a)=-1 \quad\Rightarrow\quad \boxed{a=e^{-1}=\frac{1}{e}}$

Note the solution $a=\frac{1}{e}$ is consistent with $0<a<1$.

Tie Breaker 2:

Suppose that we have a function which is continuous and has positive values on an interval [*a, b*]. We want to approximate $\int_a^b f(x)\,dx$ with the Midpoint Rule. Note that one way to visualize this approximation is to approximate the desired area with the sum of areas of rectangles.

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Next, we see that we can also visualize this same approximated area given by the Midpoint Rule as the sum of the areas of right trapezoids with the tops of the trapezoids being line segments which are tangent to the curve at the horizontal middle of each interval.

- Add these right trapezoids to your illustration.
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    B. Under
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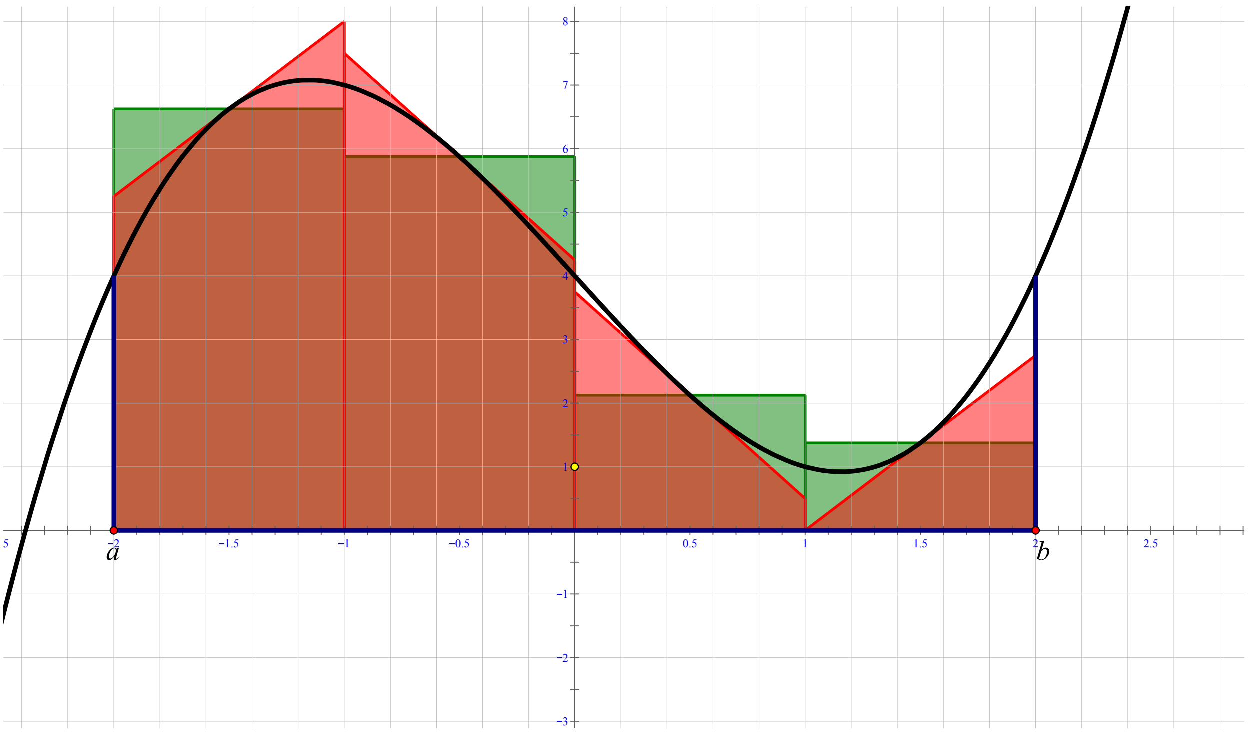


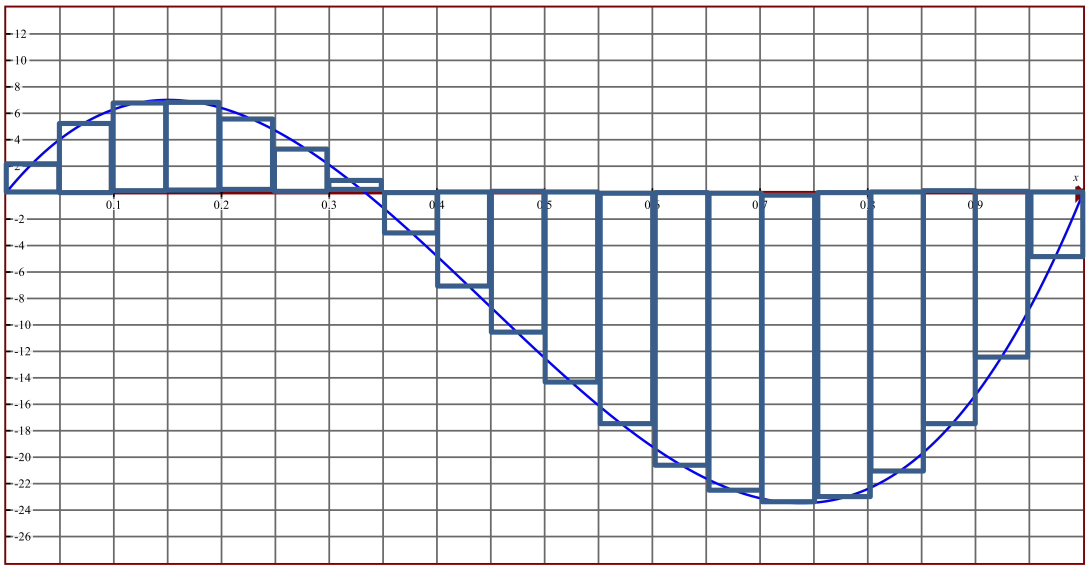
For each subinterval consider the two triangles formed. Notice that since the common point of the curve, top of the trapezoid, and top of the rectangle is at the horizontal midpoint of the interval the distance from this point to the left and right of the interval is the same, making the horizontal sides of the two triangles congruent. The pair of vertical angles formed by the top of the trapezoid and top of the rectangle are congruent, and both triangles have a right angle. Therefore, by the ASA Triangle Congruence Theorem the two triangles are congruent and thus have equal areas. So, notice that the area of the trapezoid is the same as the area of the corresponding rectangle by cutting off a triangle of the rectangle and rotating it up to form the right trapezoid. Therefore, the two illustrations of the Midpoint Rule via areas of rectangles and areas of right trapezoids produce exactly the same area.

So, the Midpoint Rule is equivalent to computing an area where the function is approximated by a tangent line segment on each subinterval. Thus, if the function is linear on each subinterval, the Midpoint Rule will be exactly the same as the actual integral. If the function is concave down on each subinterval, then the tangent line will be above the curve and the Midpoint Rule will be an over estimate of the actual integral. If the function is concave up on each subinterval then the tangent line will be below the curve and the Midpoint Rule will be an under estimate of the actual integral.

Name ________________________________

Tie Breaker 3:

A bicyclist is pedaling along a straight road with velocity, v, (in miles per hour) given in the graph below as a function of the time since starting in hours. Suppose the cyclist starts 5 miles from the lake, and that positive velocities take her *toward* the lake and negative velocities take her *away* from the lake. At the point she is farthest from the lake how far away is she?



Note that the y-values are velocities in mile/hour and the x-values are times in hours. The grid lines on the vertical v-axis come every 5 miles per hour and the gridlines on the horizontal t-axis come every 0.05 hours = 3 minutes. Therefore, areas on this graph represent distances in (mi./hr.)(hr) = miles. Areas above the t-axis represent distances traveled toward the lake and areas below the t-axis represent distances traveled away from the lake.

Since the formula for this curve was not supplied we will have to estimate these areas by drawing in rectangles, finding their areas, and summing. If we use the given gridlines the common width of the rectangles (0.05) may be factored out and we estimate the signed heights of the rectangles.

distance from the lake

$$= \left| -5 + \int_0^1 v(t)\, dt \right|$$

$$= \left| -5 + 0.05 \begin{pmatrix} 2.2 + 5.4 + 6.8 + 6.8 + 5.6 + 3.4 + 0.6 - 3.0 - 6.8 - 10.6 - 14.4 \\ -17.6 - 20.4 - 22.4 - 23.4 - 23.0 - 21.2 - 17.6 - 12.2 - 4.6 \end{pmatrix} \right|$$

$$= \left| -5 + 0.05(-166.4) \right| = \left| -5 + -8.32 \right| = \left| -13.32 \right| = 13.3$$

<u>After traveling for 1 hour she ends up at her maximum distance of 13.3 miles from the lake.</u>

[Note that there is a lot of estimating here. Answers given here are correct to one decimal place (given that the question designer knows the underlying formula for the curve). Answers using a correct method which are close to this value should be counted as correct.]