ACTM State Calculus Competition 2018

Work the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then attempt the tie-breaker questions at the end starting with tie breaker #1, then #2, and finally #3. Turn in your answer sheet and the tie-breaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Figures are not necessarily drawn to scale. Angles are in radians unless otherwise stated.

1. $\lim_{x\to 0^+}\left(\left(\tan(2x)\right)^x\right)=$
   A. 0
   B. 1
   C. 2
   D. ∞
   E. Each of the other answers is incorrect.

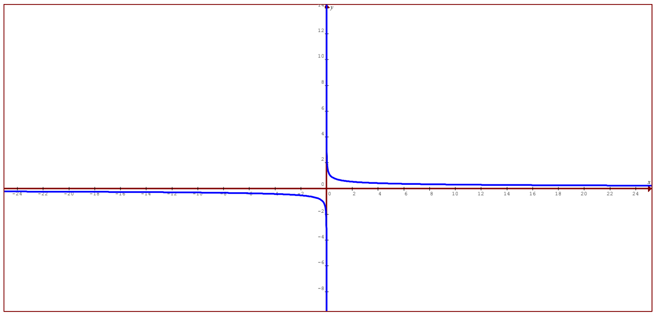
2. For extremely large natural number values of n, which of the following gives the appropriate increasing order?
   A. $\frac{1}{8}n^8 < 90n^5 < \left(\frac{5}{4}\right)^n < n^n < n!$
   B. $\left(\frac{5}{4}\right)^n < \frac{1}{2}n^8 < 90n^5 < n! < n^n$
   C. $90n^5 < \frac{1}{2}n^8 < \left(\frac{5}{4}\right)^n < n! < n^n$
   D. $90n^5 < \frac{1}{2}n^8 < \left(\frac{5}{4}\right)^n < n^n < n!$
   E. Each of the other answers is incorrect.

3. Let u and v be functions of x. Evaluate $\frac{d}{dx}\left(u^v\right)=$
   A. $vu^{v-1}\frac{du}{dx}$
   B. $vu^{v-1}\frac{dv}{dx}$
   C. $\ln(u)u^v\frac{dv}{dx}$
   D. $vu^{v-1}\frac{du}{dx}+\ln(u)u^v\frac{dv}{dx}$
   E. Each of the other answers is incorrect.

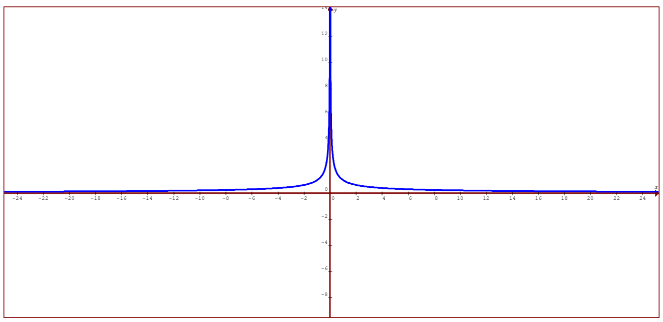
4. $f(x)=(x+3)^2(x-2)^3$ For what values of x is $f'(x)=0$?
   A. -3, 2
   B. -3, 2, 3
   C. -3, -2, 2
   D. -3, -1, 2
   E. Each of the other answers is incorrect.

5. $f(x)=x^{\frac{p}{q}}$ where $\frac{p}{q}$ is a constant reduced rational number with $0<\frac{p}{q}<1$, p is even, and q is odd. Which of the following could be a graph of $y=f'(x)$?

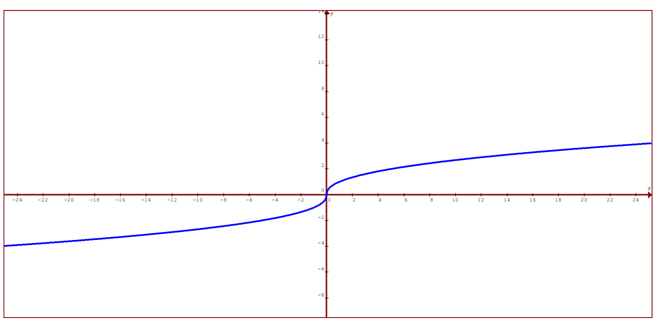
A.



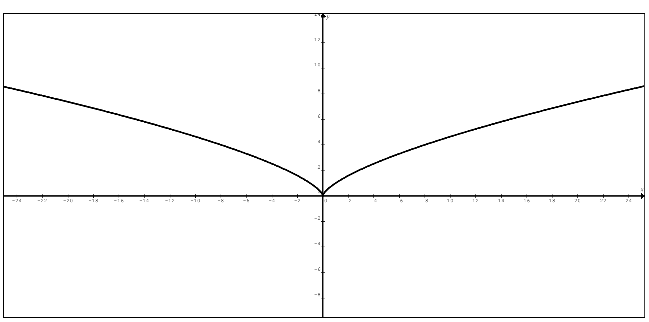
B.



C.



D.



E. Each of the other answers is incorrect.

6. If the graph of $y = f(x)$ has origin symmetry, then the graph of $y = f'(x)$ ____________________.
   A. has origin symmetry
   B. has x-axis symmetry
   C. has y-axis symmetry
   D. does not have symmetry about the origin or either coordinate axis
   E. may or may not have symmetry about the origin or a coordinate axis

7. There is a solid with a base bounded by the curves $y = 2 - 2x$ and $y = 2 - x - x^2$. Cross sections perpendicular to the x-axis are equilateral triangles. What is the volume of the solid?
   A. $\frac{\sqrt{3}}{120}$
   B. $\frac{\sqrt{3}}{60}$
   C. $\frac{1}{120}$
   D. $\frac{\sqrt{2}}{60}$
   E. Each of the other answers is incorrect.

8. Evaluate $\frac{d}{dx}\left(\tan\left(e^{3x^2+1}\right)\right) =$
   A. $6xe^{3x^2+1}\sec^2\left(e^{3x^2+1}\right)$
   B. $\sec^2\left(e^{6x}\right)$
   C. $e^{6x}\sec^2\left(e^{3x^2+1}\right)$
   D. $\sec^2\left(6xe^{3x^2+1}\right)$
   E. Each of the other answers is incorrect.

9. $f'(x) = \dfrac{3x^2}{2x^3+1}$, $f(1) = 4$ Find $f(x)$. Assume $x > 0$.
   A. $f(x) = 4 - \dfrac{6x\left(x^3-1\right)}{\left(2x^3+1\right)^2}$
   B. $f(x) = \dfrac{x^3}{\frac{1}{2}x^4 + x} + \dfrac{10}{3}$
   C. $f(x) = x^3\ln\left(2x^3+1\right) + 4$
   D. $f(x) = \frac{1}{2}\ln\left(2x^3+1\right) + 4 - \frac{1}{2}\ln(3)$
   E. Each of the other answers is incorrect.

10. When two differentiable curves intersect we define the angle at which they meet to be the smallest angle formed by their tangent lines at the point of intersection. At what angle do $f(x)=x^2$ and $g(x)=\frac{7-x^3}{6}$ meet at their intersection in the first quadrant.

A. 0
B. $\frac{\pi}{2}$
C. $\frac{\pi}{3}$
D. $\frac{\pi}{6}$
E. Each of the other answers is incorrect.

11. $f(x)=\frac{1}{e^{2x}}$, $\quad f^{(11)}(0)=$

A. -2048
B. 0
C. $\frac{1}{2048}$
D. 1
E. Each of the other answers is incorrect.

12. A probability density function (*pdf*) for a continuous distribution is a function with non-negative values such that the area between the graph of the *pdf* and the *x*-axis represents a probability, e.g. P(3 < *x* < 7) is the same as the area between the *pdf* and the *x*-axis between *x* = 3 and *x* = 7. For a particular distribution *pdf*(*x*) = 0 for negative values of *x*. What does $cdf(x)=\int_0^x pdf(x)\,dx$ give us?

A. A function whose input is the probability that the random variable is less than the output.
B. A function whose output is the probability that the random variable is greater than the input.
C. A function whose output is the probability that the random variable is less than the input.
D. A function whose input is the probability that the random variable is greater than the output.
E. Each of the other answers is incorrect.

13. At time *t* = 5 the value *V* of a certain stock is about to bottom out. Which of the following is true?

A. $V'(5) > 0$, $V''(5) > 0$
B. $V'(5) < 0$, $V''(5) > 0$
C. $V'(5) < 0$, $V''(5) < 0$
D. $V'(5) > 0$, $V''(5) < 0$
E. Each of the other answers is incorrect.

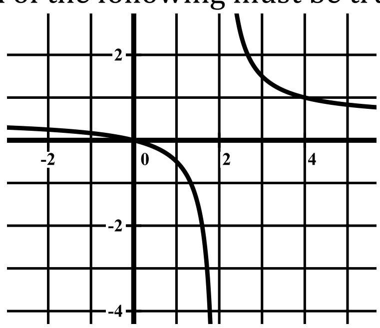
14. $\int \sin^3(x)\cos(x)\,dx=$

A. $\frac{1}{4}\cos^4(x)\sin(x)+C$
B. $-\cos^3(x)\sin(x)+C$
C. $3\cos^2(x)\sin(x)+C$
D. $-\frac{1}{4}\sin^4(x)+C$
E. Each of the other answers is incorrect.

15. On a dark night, a six-foot tall man is walking toward a 10-foot tall lamppost at a rate of 36 feet per minute. How fast is his shadow shrinking?
    A. 12 feet per minute
    B. 24 feet per minute
    C. 36 feet per minute
    D. 48 feet per minute
    E. Each of the other answers is incorrect.

16. Suppose that we approximate $\int_a^b f(x)\,dx$ with the Trapezoid and Midpoint Rules where f is decreasing and concave up on the interval $[a, b]$. Which of the following is true?
    A. The Trapezoid estimate is an over estimate and the Midpoint estimate is an over estimate.
    B. The Trapezoid estimate is an over estimate and the Midpoint estimate is an under estimate.
    C. The Trapezoid estimate is an under estimate and the Midpoint estimate is an under estimate.
    D. The Trapezoid estimate is an under estimate and the Midpoint estimate is an over estimate.
    E. Each of the other answers is incorrect.

17. The graph of f' is shown here. Which of the following must be true at x = -1.



    A. f is Increasing and Concave down.
    B. f is Decreasing and Concave up.
    C. f is Decreasing and Concave down.
    D. f is Increasing and Concave up.
    E. f is Positive.

18. A function $h(x)$ is given below. What value of k makes this function continuous over all real numbers?

$$h(x) = \begin{cases} x^3 + kx - 15 & x < 4 \\ 2x^3 - x^2 - 12x + k & x \geq 4 \end{cases}$$

    A. $k = 0$
    B. $k = 4$
    C. $k = 5$
    D. All values of k make this function continuous.
    E. No value for k can make this function continuous.

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For questions 19-20: Suppose $f(x)$ is a twice differentiable position function for a particle which is defined on the closed interval $x \in [-6, 6]$. The only critical points on the interval are given in the following table. Use the table below to answer the following two questions.

x	$f(x)$	$f'(x)$	$f''(x)$
-3	2	1	-3
-2	5	0	-2
0	1	-3	-1
2	-5	-1	2
3	-7	0	4

19. Find the total distance traveled by the particle from $x = -3$ to $x = 2$.
    A. -7
    B. -5
    C. 1
    D. 7
    E. 13

20. Which of the following must be true at $x = 4$?
    A. $f(4) > 0$
    B. $f'(4) < 0$
    C. $f'(4) > 0$
    D. $f(4) < 0$
    E. Each of the other answers is incorrect.

21. If $\int_1^8 f(x)\,dx = 3$ and $\int_8^2 f(x)\,dx = 7$ then $\int_1^2 f(x)\,dx =$
    A. -4
    B. 0
    C. 4
    D. 10
    E. 11

22. If $\int_1^8 f(x)\,dx = 4$ then $\int_1^8 \left(f(x) + 3\right)dx =$
    A. -5
    B. 1
    C. 7
    D. 20
    E. 25

23. $\lim_{x \to \frac{\pi}{6}} \left(\dfrac{\sin(x) - \sin\left(\frac{\pi}{6}\right)}{x - \frac{\pi}{6}} \right) =$

A. 0 B. $\frac{1}{2}$ C. 1 D. $\frac{\sqrt{3}}{2}$ E. Each of the other answers is incorrect.

24. Let g be a differentiable one-to-one function with the values given in the table.

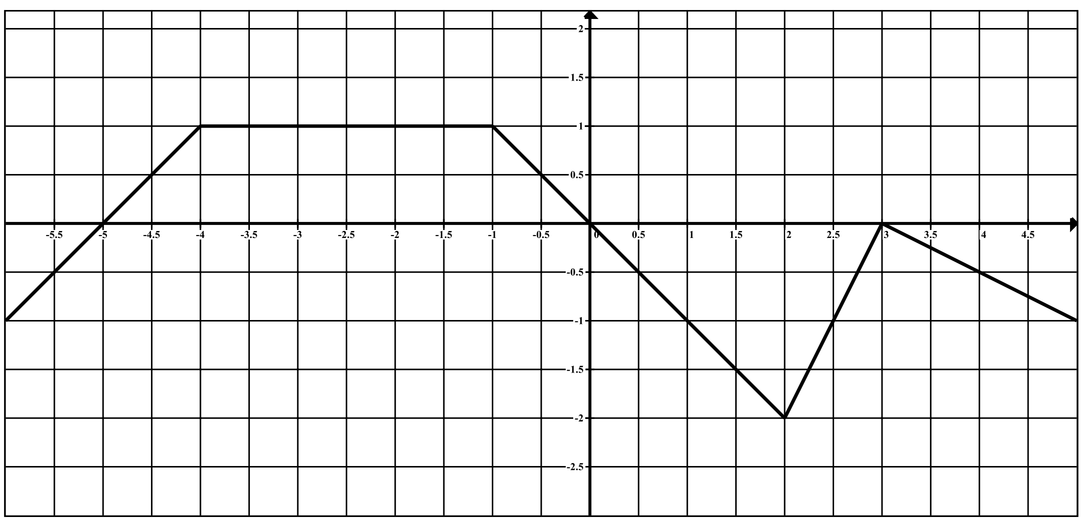
x	$g(x)$	$g'(x)$
1	4	-1
2	-6	4
3	6	-5
4	2	5

Let $h(x) = g^{-1}(x)$. $h'(2) =$

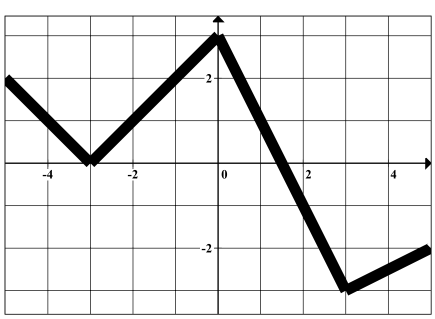
A. -5 B. -4 C. $\frac{1}{5}$ D. $\frac{1}{4}$ E. Each of the other answers is incorrect.

25. Functions f and g are given by the graphs below.

f:



g:



Let $q(x) = \dfrac{f(x)}{g(x)}$. $q'(-2) =$

A. -1
B. 0
C. 1
D. 2
E. Does not exist

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Tiebreaker Question 1

Name______________________

School______________________

Susie bought a cylindrical above ground swimming pool with diameter 24 ft and height of 4 ft. She was planning a swimming party at 6 p.m. on Saturday night and wondered when she should begin filling the pool. She decided to fill a 5-gallon bucket (5 gallons = 0.668 cubic feet). It took the bucket 5 seconds to fill up with the hose flowing at its maximum rate.

A. Find the rate of change of the volume of the bucket in cubic feet per minute.

B. Explain why the rate of change of the volume of the bucket would equal the rate of change of the volume in the swimming pool.

C. Use calculus to determine the rate of change of the depth of water in the swimming pool.

D. When would Susie need to begin filling the pool for it to be full by exactly 6 p.m. Saturday night?

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Tiebreaker Question 2

Name ____________________

School ____________________

A farmer is building a storage silo in the shape of a cylinder. The cost of the concrete floor is \$6 per square foot, the cost of the walls is \$5 per square foot, and the cost of the flat roof is \$5 per square foot. It must have a height of at least six feet. The volume is to be 4500 cubic feet. What dimensions will minimize the cost of the silo? What is this minimum cost?

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Tiebreaker Question 3

Name ____________________

School ____________________

Enter the value of the following two limits.

A. $\lim_{h \to 0} \left(\frac{\sin(h)}{h} \right) =$ ________________

B. $\lim_{h \to 0} \left(\frac{\cos(h) - 1}{h} \right) =$ ________________

Complete the following trigonometry identity.

C. $\sin(x+h) =$ ____

You do not have to provide any work or explanation for the portion above.

D. Use the information above to assist you in proving that $\frac{d}{dx}\left(\sin(x)\right) = \cos(x)$.

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Key

1	B	
2	C	
3	D	
4	D	
5	A	
6	C	
7	A	
8	A	
9	D	
10	B	
11	A	
12	C	
13	B	
14	E	$\frac{1}{4}\sin^4(x)+C$
15	E	54 feet per min
16	B	
17	A	
18	C	
19	E	
20	C	
21	D	
22	E	
23	D	
24	D	
25	A	

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Tie Breaker 1 Solution

Susie bought a cylindrical above ground swimming pool with diameter 24 ft and height of 4 ft. She was planning a swimming party at 6 p.m. on Saturday night and wondered when she should begin filling the pool. She decided to fill a 5-gallon bucket (0.668 cubic feet). It took the bucket 5 seconds to fill up with the hose flowing at its maximum rate.

A. Find the rate of change of the volume of the bucket in cubic feet per second.

$$\tfrac{dV}{dt} = \left(\frac{0.668\, ft^3}{5\sec.}\right)\left(\frac{60\sec}{1\min}\right) = \boxed{8.016\tfrac{ft^3}{\min}}$$

B. Explain why the rate of change of the volume of the bucket would equal the rate of change of the volume in the swimming pool.

Since the hose is flowing at its maximum rate when filling up the bucket it is reasonable to assume that it will continue to produce water at this same constant flow rate whether it is filling up the bucket or the pool.

C. Use calculus to determine the rate of change of the depth of water in the swimming pool.

Let t = time in minutes since beginning to fill the pool. Since the pool is in the shape of a circular cylinder we have the following relationship between the depth D of the water in the pool at time t and the volume V of water at the same time:

$$V = \pi r^2 D$$
$$V = \pi(12)^2 D$$
$$V = 144\pi D$$

Note that both V and D are changing with time. Differentiating both sides of the equation with respect to t gives us the following differential equation:

$$\frac{dV}{dt} = 144\pi\frac{dD}{dt}$$
$$\frac{dD}{dt} = \frac{1}{144\pi}\frac{dV}{dt} = \frac{1}{144\pi\ ft^2}\left(8.016\tfrac{ft^3}{\min}\right) = \frac{8.016}{144\pi}\tfrac{ft}{\min} \approx \boxed{0.1772\tfrac{ft}{\min}}$$

D. When would Susie need to begin filling the pool for it to be full by exactly 6 p.m. Saturday night?

The total volume of the full pool is $V = \pi r^2 h = \pi(12\ ft)^2(4\ ft) = 576\pi\ ft^3 \approx 1809.557\ ft^3$. So, it will take her $\left(1809.557\ ft^3\right)\left(\frac{1\min}{8.016\ ft^3}\right) \approx 225.7\min \approx 3hr46\min$ to fill the pool so she need to start filling at 6:00-3:46 = **2:14 pm** to finish filling the pool at exactly 6:00 pm.

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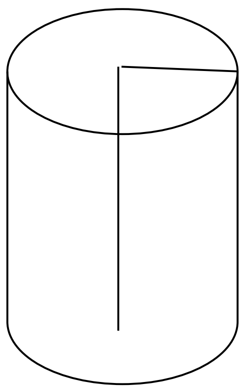
Tie Breaker 2 Solution

A farmer is building a storage silo in the shape of a cylinder. The cost of the concrete floor is \$6 per square foot, the cost of the walls is \$5 per square foot, and the cost of the flat roof is \$5 per square foot. It must have a height of at least six feet. The volume is to be 4500 cubic feet. What dimensions will minimize the cost of the silo? What is this minimum cost?



The volume is fixed at 4500 ft^3 so this gives us a relationship between the radius and height of the cylinder:

$$V = Bh = pr^2h$$

$$h = \frac{V}{pr^2} = \frac{4500}{pr^2}$$

The cost is based upon the area:

$$C = 6(floor\ area) + 5(wall\ area) + 5(roof\ area)$$

$$C = 6pr^2 + 5(2prh) + 5pr^2$$

$$C = 11pr^2 + 10prh$$

$$C = 11pr^2 + 10pr\left(\frac{4500}{pr^2}\right)$$

$$\underline{C(r) = 11pr^2 + 45000r^{-1}}$$

$$\Rightarrow \underline{C'(r) = 22pr - 45000r^{-2}}$$

$$C'(r) = 0 = 22pr - 45000r^{-2}$$

$$r^2(0) = r^2\left(22pr - 45000r^{-2}\right)$$

$$0 = 22pr^3 - 45000$$

$$45000 = 22pr^3$$

$$r^3 = \frac{22500}{11p}$$

$$r = \sqrt[3]{\frac{22500}{11p}} \approx 8.66722331994$$

Editor's Note:
$p = \pi$

Therefore, the cost is minimized when the radius is approximately 8.667 feet and the height is 19.068 feet. At this point the cost is \$7,787.96.

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Tie Breaker 3 Solution

Enter the value of the following two limits.

A. $\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right) = \boxed{1}$ B. $\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) = \boxed{0}$

Complete the following trigonometry identity.

C. $\sin(x+h) = \boxed{\cos(x)\sin(h)+\sin(x)\cos(h)}$

You do not have to provide any work or explanation for the portion above.

D. Use the information above to assist you in proving that $\frac{d}{dx}(\sin(x)) = \cos(x)$.

Proof :

$$\frac{d}{dx}(\sin(x)) = \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin(x)}{h}\right)$$ definition of derivative

$$= \lim_{h\to 0}\left(\frac{\cos(x)\sin(h)+\sin(x)\cos(h)-\sin(x)}{h}\right)$$ identity C above

$$= \lim_{h\to 0}\left(\frac{\cos(x)\sin(h)+\sin(x)(\cos(h)-1)}{h}\right)$$ factoring via distributive property

$$= \lim_{h\to 0}\left(\frac{\cos(x)\sin(h)}{h}+\frac{\sin(x)(\cos(h)-1)}{h}\right)$$ distributive property

$$= \lim_{h\to 0}\left(\frac{\cos(x)\sin(h)}{h}\right) + \lim_{h\to 0}\left(\frac{\sin(x)(\cos(h)-1)}{h}\right)$$ limit of a sum is sum of limits if they exist

$$= \cos(x)\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right) + \sin(x)\lim_{h\to 0}\left(\frac{(\cos(h)-1)}{h}\right)$$ limit of a constant times a function is the constant times the limit of the function

$$=\cos(x)(1) + \sin(x)(0)$$ limit properties A and B avove

$$=\cos(x)$$ properties of multiplying by 1 and 0