Arkansas Council of Teachers of Mathematics
2019 Algebra II State Exam

Work the multiple choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then, attempt the tie breaker questions at the end starting with Tie Breaker #1, then #2, and finally #3. Turn in your answer sheet and the tie breaker pages when you are finished. You may keep the pages with the multiple choice questions.

1. Find a polynomial of degree four with real coefficients that has the complex roots $1 + i$ and $2 + 2i$.

    A. $x^4 - 6x^3 + 18x^2 - 24x + 16$
    B. $2x^4 + 16x^3 - 5$
    C. $x^4 - 24x^3 + 16x^2 - 18x + 7$
    D. $5x^4 + 11x^2 - 23x + 4$
    E. No polynomial is possible

2. Victor's age plus the cube of Zoe's age is 1,739. Zoe's age plus the cube of Victor's age is 1,343. How old are Victor and Zoe?

    A. Zoe is 3 and Victor is 7
    B. Zoe is 12 and Victor is 11
    C. Zoe is 5 and Victor is 11
    D. Zoe is 13 and Victor is 9
    E. Unable to determine

3. Determine the real numbers D, E, and F so that the following equation is an identity.
$$D(2x + 1) + (Ex + F)(2x - 1) = (x + 1)(4x + 3)$$

    A. $D = 3/2\,, E = 7, F = -2$
    B. $D = -1, E = -1, F = 0$
    C. $D = 0, E = 1, F = -1/2$
    D. $D = 15/4\,, E = 2, F = 3/4$
    E. Unable to determine

4. Find the palindromic number between 50,000 and 60,000 which is divisible by 9 with no remainder, and whose hundreds place is 0.

   A. 59095
   B. 54054
   C. 54045
   D. 51015
   E. No value exists

5. Find the quotient $q(x)$ and the remainder $r(x)$ if $f(x) = 2x^3 - x^2 + 1$ is divided by $g(x) = 3x^4 - 7$.

   A. $q(x) = x + 7,\ r(x) = x^2 - 3x$
   B. $q(x) = 0,\ r(x) = 2x^3 - x^2 + 1$
   C. $q(x) = x^2 - 2,\ r(x) = -3x + 1$
   D. $q(x) = 2x^3 - x^2 + 1,\ r(x) = -1$
   E. Unable to determine

6. Anastasia likes to call her friend Daphne in Guam from her home in Arkansas. Anastasia's mom makes her pay for all her out of country phone calls. Last Sunday, Anastasia called Daphne at 7:00 a.m. and ended the phone conversation at 8:30 a.m. Before 8:00 a.m. on Sundays, it costs $0.35 for the first minute and then $0.20 per minute after that to make the call. After 8:00 a.m., the rate goes up to $0.40 for the first minute and $0.25 per minute after that. How much does Anastasia owe her mom for the phone call?

   A. $14.70
   B. $17.35
   C. $19.65
   D. $23.50
   E. Unable to determine

7. The two complex roots of the equation $(x-4)(x-5)(x-6)(x-7)=1680$ are:

   A. $\frac{11\pm\sqrt{159}i}{2}$
   B. $\frac{1\pm\sqrt{3}i}{2}$
   C. $\frac{3\pm\sqrt{153}i}{2}$
   D. $\frac{13\pm\sqrt{97}i}{3}$
   E. There are no complex roots

8. Define the function $\delta(r+1)=r\delta(r)$ where $\delta(1)=1$ when r is a positive integer. Find $\delta(7)$.

   A. 120
   B. 24
   C. 5040
   D. 720
   E. Unable to determine

9. Find the set of values for x such that $x^3+1>x^2+x$.

   A. $|x|\le 1$ or $x>1$
   B. $0<x<1$ or $x>1$
   C. x is any real number
   D. $|x|<1$ or $x>1$
   E. Unable to determine

10. One of the solutions to the following equation is $x = 3$. Find the other real number solution(s) of x.

$$3^{2x^2-7x+3} = 4^{x^2-x-6}$$

A. $x = \frac{1+2\left(\frac{\log 4}{\log 3}\right)}{2-\left(\frac{\log 4}{\log 3}\right)}$
B. $x = 1 - \frac{\log 4}{\log 3}$
C. $x = \frac{1-2\left(\frac{\log 4}{\log 3}\right)}{1+\left(\frac{\log 4}{\log 3}\right)}$
D. $x > \log 3 - \log 4$
E. No value for x exists

11. Given the three equations $7x - 12y = 42$, $7x + 20y = 98$, and $21x + 12y = m$, find all the value(s) of m in which these three lines intersect at a single common coordinate.

A. $m = \pm 71, 142$
B. $m = 24, 48$
C. $m = 210$
D. $m = \pm 101$
E. No value for m exists

12. Express the following equation with a rational denominator.

$$Q = \frac{\sqrt[3]{2}}{1 + 5\sqrt[3]{2} + 7\sqrt[3]{4}}$$

A. $Q = \frac{23\sqrt[3]{2}-31\sqrt[3]{4}+17}{141}$
B. $Q = \frac{-23\sqrt[3]{2}+31\sqrt[3]{4}+12}{471}$
C. $Q = \frac{-2\sqrt[3]{2}+\sqrt[3]{4}}{471}$
D. $Q = \frac{-29\sqrt[3]{2}+23\sqrt[3]{4}-15}{141}$
E. No rational denominator is possible

13. Find the real values of x such that $x \log_2 3 = \log_{10} 3$. Round to three decimal places.

    A. $x \approx 1.311$
    B. $x \approx 0.301$
    C. $x \approx 3.011$
    D. $x \approx 0.103$
    E. No value for x exists.

14. If q, r, and s are three consecutive odd integers such that $q < r < s$ then find the value of $q^2 - 2r^2 + s^2$.

    A. 7
    B. 9
    C. 10
    D. 8
    E. No value exists

15. A woman sells a flux capacitor for \$171.00 gaining on the sale as many percent (based on the cost) as the cost of the flux capacitor P. Find P in dollars.

    A. \$90.00
    B. \$92.00
    C. \$85.00
    D. \$104.00
    E. Not enough information given

16. Find the simplest form: $W = \sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$.

    A. $\sqrt{6}$
    B. $3\sqrt{2}$
    C. $\sqrt{11}$
    D. 1
    E. Already in simplest form

17. Olivia and Amelia are sisters who share a room. Olivia can clean the room by herself in 2 hours. If Amelia helps, they can clean the room together in 1 hour and 15 minutes. How long would it take just Amelia to clean the room by herself?

    A. 45 minutes
    B. 1 hour
    C. 3 hours 15 minutes
    D. 3 hours 20 minutes
    E. Unable to determine

18. Identify the extraneous solution to the following equation.

$$\sqrt{x+3} = x - 9$$

    A. $x = -3$
    B. $x = 6$
    C. $x = 13$
    D. $x = \frac{1-\sqrt{337}}{2}$
    E. Unable to determine

19. Given that $2^{3x} = 16^{y+1}$ and $2x = 5y - 17$, find the value of $x + y$.

    A. 18
    B. 21
    C. 40
    D. 12
    E. Unable to solve for x and y

20. If $\log_4 x = -\frac{3}{2}$, solve for x.

A. $\frac{1}{8}$
B. $\frac{1}{64}$
C. -6
D. $-\frac{1}{6}$
E. Unable to solve for x

21. Factor the following expression completely: $6x^3 - 4x^2 - 16x$

A. $2x(3x^2 - 2x - 8)$
B. $2x(3x + 4)(x - 2)$
C. $4x(2x + 1)(x - 4)$
D. $2x(2x^2 + 7x - 4)$
E. Not factorable

22. If N is the sum of three consecutive 2-digit prime numbers, and N is also the product of two consecutive 2-digit prime numbers, what is the least possible value of N?

A. 97
B. 121
C. 143
D. 150
E. No value for N exists

23. Completely factor the following sum of cubes: $a^3 + b^3$

A. $(a + b)(a^2 + ab + b^2)$
B. $(a + b)(a^2 + ab - b^2)$
C. $(a - b)(a^2 + ab + b^2)$
D. $(a + b)(a^2 - ab + b^2)$
E. Not factorable

24. Divide the complex number $3 + 2i$ by the complex number $4 - 3i$, and provide the resulting quotient in complex number form.
    A. $\frac{6}{25} + \frac{17}{25}i$
    B. $\frac{23}{25} + \frac{16}{25}i$
    C. $1 + \frac{6}{25}i$
    D. $\frac{19}{20} - \frac{7}{20}i$
    E. Not possible to divide

25. Two consecutive odd numbers are such that three times the first is 5 more than twice the second. What are those two odd numbers?

    A. 5 & 7
    B. 7 & 9
    C. 9 & 11
    D. 11 & 13
    E. No numbers exist

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Tie Breaker #1

Name: ___________________

School: _________________

Find all solutions to the following equation. Provide exact values.

$$x^4 + x^3 - 5x^2 - 15x - 18 = 0$$

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Tie Breaker #2

Name: ___________________

School: __________________

Given the following equations:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

Show that the following identity is true:

$$\omega^2 - \omega_0^2 = 2\alpha\theta$$

Tie Breaker #3

Name: ___________________

School: __________________

The golden ratio is the name of the value $\varphi = \frac{1+\sqrt{5}}{2}$. It is a solution to the equation $\varphi^2 = \varphi + 1$. Using this value, find the infinite sum of the reciprocal powers of φ. That is, evaluate the following for S. Give an exact value for your result.

$$S = 1 + \frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} + \frac{1}{\varphi^5} + \frac{1}{\varphi^6} + \cdots$$

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Multiple Choice Answer Key

1. A
2. B
3. D
4. C
5. B
6. C
7. A
8. D
9. D
10. A
11. C
12. B
13. B
14. D
15. A
16. A
17. D
18. B
19. B
20. A
21. B
22. C
23. D
24. A
25. C

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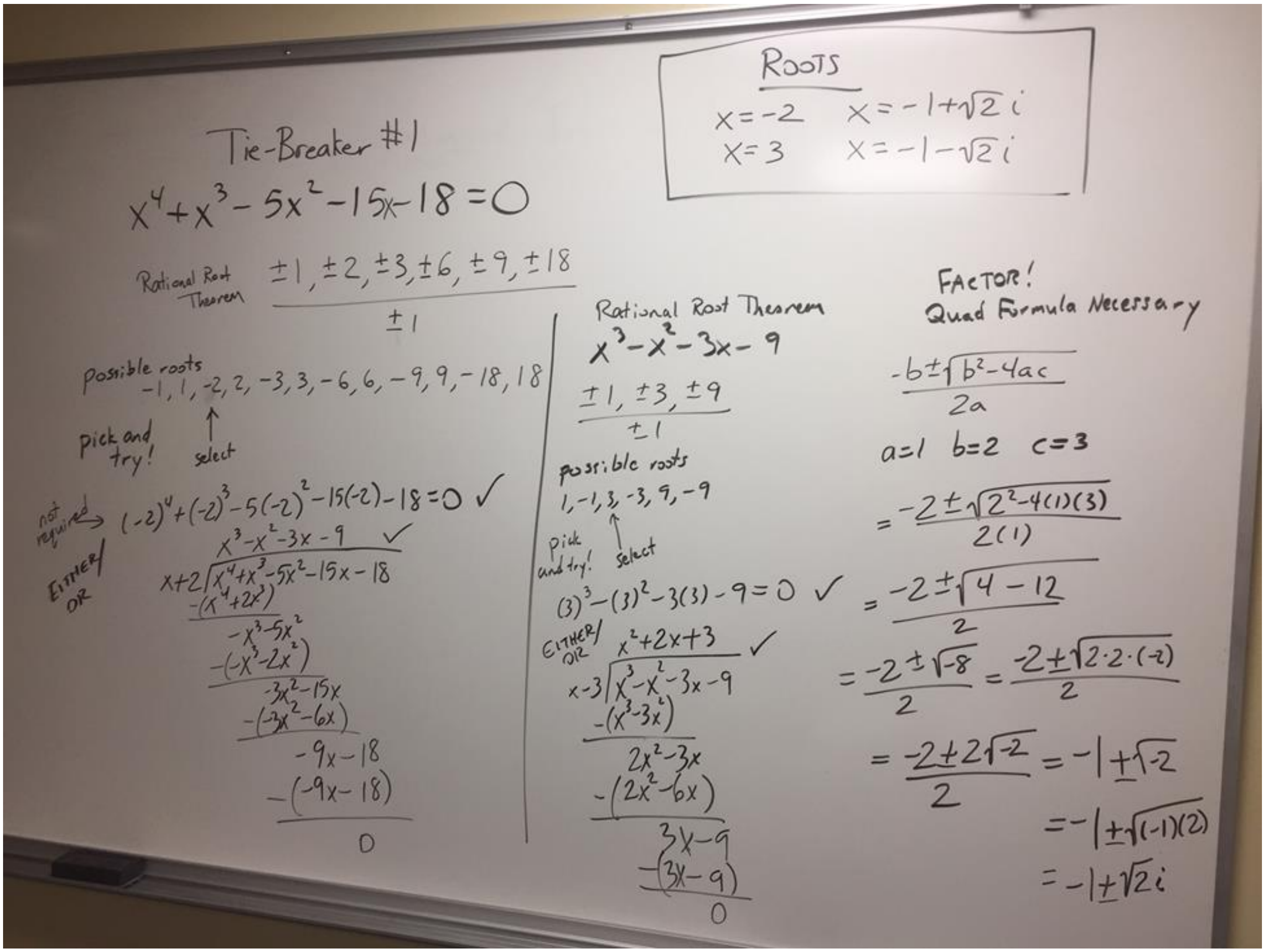
Tie Breaker #1

Find all solutions to the equation $x^4 + x^3 - 5x^2 - 15x - 18 = 0$.

There are a few options to work this. One is shown below.



Tie Breaker #2

Given the following equations: $\omega = \omega_0 + \alpha t$ and $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

Show that the following identity is true: $\omega^2 - \omega_0^2 = 2\alpha\theta$

① $w = w_0 + \alpha t$
② $\theta = w_0 t + \frac{1}{2}\alpha t^2$ } given $\Rightarrow$ ① $w - \alpha t = w_0$
$w - w_0 = \alpha t$

Tie-Breaker #2

Show $w^2 - w_0^2 = 2\alpha\theta$ (*)

Computing the LHS of (*) we have

$(w_0 + \alpha t)^2 - (w - \alpha t)^2$

$= w_0^2 + 2\alpha t w_0 \cancel{+ \alpha^2 t^2} - w^2 + 2\alpha t w \cancel{- \alpha^2 t^2}$

$= w_0^2 - w(w - 2\alpha t) + 2\alpha t w_0$

$= w_0^2 - (w_0 + \alpha t)(w - \alpha t - \alpha t) + 2\alpha t w_0$

$= w_0^2 - (w_0 + \alpha t)(w_0 - \alpha t) + 2\alpha t w_0$

$\left[\begin{array}{l} w_0 = w - \alpha t \\ w = w_0 + \alpha t \end{array} \right.$

$= \cancel{w_0^2 - w_0^2} + \alpha^2 t^2 + 2\alpha t w_0$

$= 2\alpha(w_0 t + \frac{1}{2}\alpha t^2)$

$= 2\alpha\theta$

which the RHS of (*)

$= 2\alpha\theta$

which is the RHS of (*)

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Tie Breaker #3

The golden ratio is defined as the value $\varphi = \frac{1+\sqrt{5}}{2}$. It is a solution to the equation $\varphi^2 = \varphi + 1$. Using this value, find the infinite sum of the reciprocal powers of φ. That is, evaluate the following for S. Give an exact value for your result.

$$S = 1 + \frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} + \frac{1}{\varphi^5} + \frac{1}{\varphi^6} + \cdots$$

Solution:
It is given that

$$S = 1 + \frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} + \frac{1}{\varphi^5} + \frac{1}{\varphi^6} + \cdots$$

We can also calculate

$$\begin{aligned} S \cdot \varphi &= \left(1 + \frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} + \frac{1}{\varphi^5} + \frac{1}{\varphi^6} + \cdots\right) \cdot \varphi \\ &= \varphi + 1 + \frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} + \frac{1}{\varphi^5} + \frac{1}{\varphi^6} + \cdots \end{aligned}$$

Then calculate the difference between the two.

$$\begin{aligned} S - S \cdot \varphi &= \left(1 + \frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} + \frac{1}{\varphi^5} + \frac{1}{\varphi^6} + \cdots\right) - \left(\varphi + 1 + \frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} + \frac{1}{\varphi^5} + \frac{1}{\varphi^6} + \cdots\right) \\ &= -\varphi + 1 - 1 + \frac{1}{\varphi} - \frac{1}{\varphi} + \frac{1}{\varphi^2} - \frac{1}{\varphi^2} + \frac{1}{\varphi^3} - \frac{1}{\varphi^3} + \frac{1}{\varphi^4} - \frac{1}{\varphi^4} + \frac{1}{\varphi^5} - \frac{1}{\varphi^5} + \frac{1}{\varphi^6} - \frac{1}{\varphi^6} + \cdots \\ &= -\varphi \end{aligned}$$

Now solve the equation $S - S \cdot \varphi = -\varphi$ for S.

$$\begin{aligned} S &= \frac{-\varphi}{1-\varphi} = \frac{-\left(\frac{1+\sqrt{5}}{2}\right)}{1 - \left(\frac{1+\sqrt{5}}{2}\right)} = \left(-\frac{1+\sqrt{5}}{2}\right) \div \left(1 - \frac{1+\sqrt{5}}{2}\right) \\ &= \left(-\frac{1+\sqrt{5}}{2}\right) \div \left(\frac{2}{2} - \frac{1+\sqrt{5}}{2}\right) = \left(-\frac{1+\sqrt{5}}{2}\right) \div \left(\frac{2-1-\sqrt{5}}{2}\right) \\ &= \left(-\frac{1+\sqrt{5}}{2}\right) \div \left(\frac{1-\sqrt{5}}{2}\right) = \left(-\frac{1+\sqrt{5}}{2}\right) \cdot \left(\frac{2}{1-\sqrt{5}}\right) = -\frac{1+\sqrt{5}}{1-\sqrt{5}} \\ &= -\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right) \cdot \left(\frac{1+\sqrt{5}}{1+\sqrt{5}}\right) = -\frac{\left(1+2\sqrt{5}+5\right)}{1-5} = -\frac{6+2\sqrt{5}}{-4} \\ &= \frac{3+\sqrt{5}}{2} \end{aligned}$$