Arkansas Council of Teachers of Mathematics
2019 Calculus State Exam

Work the multiple choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then, attempt the tie breaker questions at the end starting with Tie Breaker #1, then #2, and finally #3. Turn in your answer sheet and the tie breaker pages when you are finished. You may keep the pages with the multiple choice questions.

Figures aren't necessarily drawn to scale. Angles are given in radians unless otherwise stated. Assume all constants are real numbers.

1. Let $f(x)$ be a differentiable function on $(-\infty, \infty)$ such that $f(x)$ has a relative maximum of 6 at $x = 2$. Let $g(x) = x \cdot f(x) + x^2$. Evaluate $g'(2)$.

   A. 0
   B. 10
   C. -6
   D. 12
   E. None of the above

2. Determine the equation of the tangent line to the curve $xy^2 + x = 3y + 1$ at $(1,4)$.

   A. $y = -\frac{1}{5}x - \frac{21}{5}$
   B. $y = -\frac{2}{5}x + \frac{13}{5}$
   C. $y = -\frac{17}{5}x + \frac{37}{5}$
   D. $y = \frac{17}{5}x + \frac{7}{5}$
   E. None of the above

3. Evaluate: $\frac{d}{dx}\int_{x^2}^{0} \sin t^2 \, dt =$

   A. $x^2 \sin 2x$
   B. $-2x \sin x^4$
   C. $-4x^3 \cos x^4$
   D. $2x \sin x^2$
   E. None of the above

4. Let a be a constant and let f be a continuous function such that $f(a+x) = -f(a-x)$ for all x. Then, evaluate $\int_{a-2}^{a+2} f(x)\,dx$.

   A. 0
   B. $2a + 4$
   C. $\frac{1}{2}a$
   D. 4
   E. None of the above

5. Let $f(x) = 3^{2x^2+3x}$. Find $f'(x)$.

   A. $(\ln 3)3^{4x+3}$
   B. $(\ln 3)(4x+3)3^{2x^2+3x}$
   C. $(4x+3)(3^{2x^2+3x})$
   D. $\ln(4x+3)\left(3^{2x^2+3x}\right)$
   E. None of the above

6. For a positive integer $n \geq 2$, find the area of the region bounded by $f(x) = x + 3$ and $g(x) = x^{1/n} + 3$ for $x \geq 0$.

   A. $\frac{1-n}{2(n+1)}$
   B. $\frac{n}{2n+3}$
   C. $\frac{2^{n+1}}{n}$
   D. $-\frac{2n}{n+1}$
   E. None of the above

7. Find the absolute minimum value for the function $f(x) = 2x^4 + 2x^3 - 4x^2 - 6x - 2$ on the interval $[-1, 0]$.

   A. $y = -2$
   B. $y = 0$
   C. $y = -6$
   D. $y = -8$
   E. None of the above

8. Let b and c be constants such that $f(x) = \frac{2x^2+bx+c}{x-3}$. Determine a value of the constant b such that $\lim_{x \to 3} f(x) = 8$.

   A. $b = 0$
   B. $b = -2$
   C. $b = 2$
   D. $b = -4$
   E. None of the above

9. Evaluate the following, where b is a positive constant.

$$\lim_{x \to -\infty} \frac{2x^3 + 1}{\sqrt{bx^6 + 4}} =$$

   A. $-\frac{1}{\sqrt{b}}$
   B. $-\frac{2}{\sqrt{b}}$
   C. $-2b$
   D. $\frac{2}{b}$
   E. None of the above

10. Use the definite integral to find the total area between $f(x) = 3e^x - 4$ and the x-axis over the interval $[-3,3]$. Round your answer to one decimal place.

   A. 54.7
   B. 36.1
   C. 42.9
   D. 60.1
   E. None of the above

11. Evaluate the following, where a is a constant.

$$\lim_{x \to 0}(1 + ax)^{3/x} =$$

   A. $3a$
   B. ∞
   C. e^{3a}
   D. e^3
   E. None of the above

12. Let $f(x) = x^2 - 2kx + k^2$, where k is a constant. Determine the equation of the line tangent to $f(x)$ at $x = 0$.

   A. $y = kx + 1$
   B. $y = -k^2x + 2k$
   C. $y = -2kx + k^2$
   D. $y = 2(x + k)$
   E. None of the above

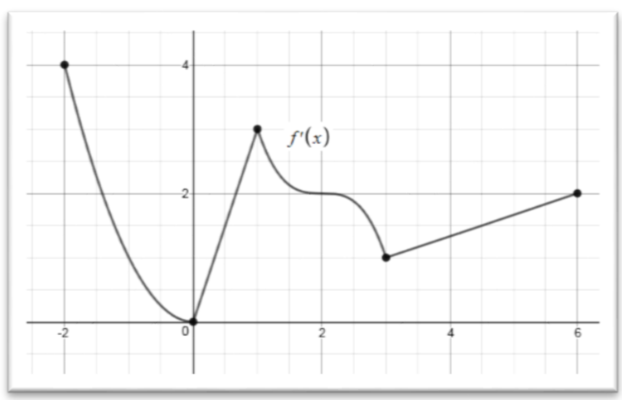
13. Suppose a ladybug is crawling along the y-axis with its position at time t given by $s(t) = -t^2 + (a+b)t - ab$, where a and b are constants. At what time, if any, is the ladybug at rest?

A. $t = \frac{a+b}{2}$
B. $t = 2a + b$
C. $t = \frac{b-a}{2}$
D. $t = a - \frac{b}{2}$
E. None of the above

14. Use a differential to approximate the change in the volume of a right circular cone of fixed height $h = 5m$ when its radius increases from $r = 4m$ to $r = 4.02m$. (Recall: $V = \frac{1}{3}\pi r^2 h$).

A. $\frac{4\pi}{15}\ m^3$
B. $0.02\ m^2$
C. $\frac{10}{3}\pi\ m^3$
D. $\frac{4}{3}\pi\ m^3$
E. None of the above

15. Using the graph of $f'(x)$ on the right, determine the interval(s) over which $f(x)$ is concave up.



A. $(2,1), (3,6)$
B. $(0,1), (3,6)$
C. $(-2,2)$
D. $(-2,0), (1,3)$
E. None of the above

16. Let $g(x) = \begin{cases} 4x - 1 & x \leq 1 \\ 3 + 4x & x > 1 \end{cases}$. Which of the following statements are true?

1. $\lim\limits_{h \to 0^-} \frac{g(1+h) - g(1)}{h} = 4$
2. $\lim\limits_{h \to 0^+} \frac{g(1+h) - g(1)}{h} = 4$
3. $g'(1) = 4$

A. 1 only
B. 1, 2, and 3
C. 1 and 2 only
D. 2 only
E. None of the above

17. Suppose a rectangle is constructed such that one base lies on the diameter of a semicircle with radius $k\ cm$ and two vertices lie on the semicircle. What are the dimensions of the rectangle with maximum area?

A. $\frac{k}{\sqrt{2}}\ cm\ \times \frac{k}{\sqrt{2}}\ cm$
B. $\frac{2k}{\sqrt{2}}\ cm\ \times \frac{k}{\sqrt{2}}\ cm$
C. $2k\ cm\ \times k\ cm$
D. $\frac{1}{\sqrt{2}}\ cm\ \times \frac{2}{\sqrt{2}}\ cm$
E. None of the above

18. Let $f(x) = \ln(sin^2(2x^3)) + 3^{\sqrt{x}}$. Find $f'(x)$.

A. $12x\cot(2x^3) + \frac{3^{\sqrt{x}}\ln 3}{2\sqrt{x}}$
B. $6x\sin(2x^3) + \frac{3^{\sqrt{x}}}{2}$
C. $-12x\cos(2x^3) + \frac{3^{\sqrt{x}}\ln 3}{2\sqrt{x}}$
D. $\frac{1}{4\sin(6x^2)} + \frac{3^{\sqrt{x}}\ln 3}{2\sqrt{x}}$
E. None of the above

19. Suppose f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). What additional condition must be satisfied to guarantee that there exists at least one point c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$?

    A. $f(b) = f(a)$
    B. f must have a relative extreme value at c
    C. $a > 0$
    D. $f(x) > 0$ for all x in (a, b)
    E. None of the above

20. Use the fact that $\int_0^{\pi/2} (\cos x - 2\sin x)\, dx = -1$ to evaluate $\int_{\pi/2}^{0} (k \sin x - \frac{k}{2} \cos x)\, dx$ for some constant k.

    A. $\frac{k}{2}$
    B. $-\frac{k}{2}$
    C. $-2k$
    D. k
    E. None of the above

21. Which of the following differentiable functions, if any, has arclength given by $\int_a^b \sqrt{1 + k\sin^2(2x)}\, dx$ on the interval $[a, b]$ for constant k?

    A. $f(x) = \frac{k}{4}(\cos^2(2x))$
    B. $f(x) = -\sqrt{k}(\sin 2x)$
    C. $f(x) = -\frac{\sqrt{k}}{2}(\cos 2x)$
    D. $f(x) = -\frac{\sqrt{k}}{2}(\sin 2x)$
    E. None of the above

22. Evaluate: $\int (3x+3)e^{7x^2+14x}dx$

A. $\frac{1}{14}e^{7x^2+14x}+C$
B. $3\ln(7x^2+14x)+C$
C. $\frac{3}{14}e^{7x^2+14x}+C$
D. $3(14x+14)e^{7x^2+14x}+C$
E. None of the above

23. A rock is thrown into the air. The height (in feet) of the rock after t seconds is given by the function $s(t)=-16t^2+96t$. Find the average height of the rock over the entire flight.

A. 21 feet
B. 32 feet
C. 64 feet
D. 96 feet
E. None of the above

24. Find the volume of the solid formed by revolving the region bounded by $y=7\csc x$ and $y=7\sqrt{2}$, $\frac{\pi}{4}\le x\le\frac{3\pi}{4}$, about the x-axis.

A. $7\pi^2-49\pi$
B. $49\pi^2+98\pi$
C. $49\pi^2-98\pi$
D. $\pi^2+14\pi$
E. None of the above

25. Evaluate $\int \sin^3(3x)\cos(3x)\,dx$.

A. $\frac{1}{12}\sin^4(3x)+C$
B. $\frac{1}{4}\sin(3x)\cos^2(3x)+C$
C. $\frac{1}{12}\sin^4(3x)\cos^2(3x)+C$
D. $\frac{1}{8}\sin^4(3x)\cos^2(3x)+C$
E. None of the above

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Tie Breaker #1

Name: ___________________

School: __________________

Let a and b be constants such that $0 < b < a$. Find the area of the region R bounded by the graphs of $y_1 = e^{-ax}$ and $y_2 = e^{-bx}$ for $x \geq 0$.

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Tie Breaker #2

Name: ___________________

School: __________________

Suppose Jacob has 100 feet of fencing with which to build two separate pens. The first is to be shaped like a circle and the second is to be shaped like a square. How much fencing material should be used to build the circular pen if the area of both pens is to be maximized? (Round your final answer to two decimal places as needed.)

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Tie Breaker #3

Name: ___________________

School: __________________

An equilateral triangle initially has sides of length $20ft$ when each vertex moves toward the midpoint of the opposite side at a rate of 1.5 ft/min. Assuming the triangle remains equilateral, what is the rate of change of the area of the triangle when the length of each side is 1 ft?

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Answer Key

1. B
2. C
3. B
4. A
5. B
6. E
7. A
8. D
9. B
10. A
11. C
12. C
13. A
14. A
15. B
16. C
17. B
18. E
19. E
20. B
21. C
22. C
23. D
24. C
25. A

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Solution Tie Breaker #1

Let a and b be constants such that $0 < a < b$. Find the area of the region R bounded by the graphs of $y_1 = e^{-ax}$ and $y_2 = e^{-bx}$ for $x \geq 0$.

Solution:

Since $0 < a < b$, we know that $e^{-bx} < e^{-ax}$ when $x \geq 0$.

We need to calculate the area for $x \geq 0$.

$$\int_0^{\infty} (e^{-ax} - e^{-bx})\,dx = \lim_{c\to\infty}\left(\int_0^{c} (e^{-ax} - e^{-bx})\,dx\right)$$

$$= \lim_{c\to\infty}\left(\left(-\frac{1}{a}e^{-ax} + \frac{1}{b}e^{-bx}\right|_0^c\right)$$

$$= \lim_{c\to\infty}\left(\left(-\frac{1}{a}e^{-ac} + \frac{1}{b}e^{-bc}\right) - \left(-\frac{1}{a}e^{-a\cdot 0} + \frac{1}{b}e^{-b\cdot 0}\right)\right)$$

$$= \lim_{c\to\infty}\left(-\frac{1}{a}e^{-ac} + \frac{1}{b}e^{-bc} + \frac{1}{a} - \frac{1}{b}\right)$$

$$= \lim_{c\to\infty}\left(\left(-\frac{1}{a} + \frac{1}{b}\right)e^{-bc} + \frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{1}{a} - \frac{1}{b}$$

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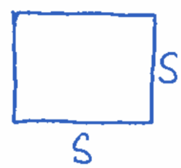
Solution Tie Breaker #2

Suppose Jacob has 100 feet of fencing with which to build two separate pens. The first is to be shaped like a circle and the second is to be shaped like a square. How much fencing material should be used to build the circular pen if the area of both pens is to be maximized? (Round your final answer to two decimal places as needed.)

Let x = Circumference of circular pen $\qquad 0 \le x \le 100$

$100 - x$ = perimeter of square pen



$x = 2\pi r$ $\qquad\qquad 4s = 100 - x$

$r = \dfrac{x}{2\pi}$ $\qquad\qquad s = \dfrac{100 - x}{4} = 25 - \dfrac{1}{4}x$

$\text{Area} = \pi r^2 + s^2$

$A(x) = \pi\left(\dfrac{x}{2\pi}\right)^2 + \left(25 - \dfrac{1}{4}x\right)^2$

$= \dfrac{1}{4\pi}x^2 + 625 - \dfrac{25}{2}x + \dfrac{1}{16}x^2$

$\left(\dfrac{1}{4\pi} + \dfrac{1}{16}\right)x^2 - \dfrac{25}{2}x + 625$

$A'(x) = 2\left(\dfrac{1}{4\pi} + \dfrac{1}{16}\right)x - \dfrac{25}{2}$

$2\left(\dfrac{1}{4\pi} + \dfrac{1}{16}\right)x - \dfrac{25}{2} = 0$

$x = 43.99$

A is a cts. fnc. on a closed interval.

$A(0) = 625$

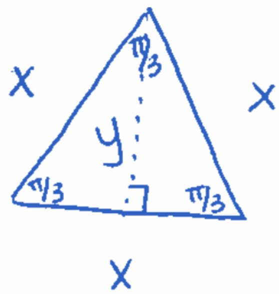
$A(100) = 795.77$

$A(43.99) = 350.06$

The area will be maximized when all 100 ft of fencing material is used for the circular pen.

Tie Breaker #3

An equilateral triangle initially has sides of length $20ft$ when each vertex moves toward the midpoint of the opposite side at a rate of $1.5\ ft/min$. Assuming the triangle remains equilateral, what is the rate of change of the area of the triangle when the length of each side is $1\ ft$?



$$\frac{dy}{dt} = -1.5\ ft/min$$

$$\sin(\pi/3) = \frac{y}{x} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow x = \frac{2}{\sqrt{3}}y$$

When $x = 1$

$$1 = \frac{2}{\sqrt{3}}y$$

$$y = \frac{\sqrt{3}}{2}$$

$$\text{Area} = \frac{1}{2}xy$$

$$A(y) = \frac{1}{2}\left(\frac{2}{\sqrt{3}}y\right)y$$

$$= \frac{1}{\sqrt{3}}y^2$$

$$\frac{dA}{dt} = \frac{2}{\sqrt{3}}y\frac{dy}{dt}$$

$$\frac{dA}{dt} = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right)(-1.5)$$

$$\boxed{\frac{dA}{dt} = -1.5\ ft^2/min}$$