ACTM Regional Calculus Competition 2018

Work the multiple-choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then attempt the tie-breaker questions at the end starting with tie breaker #1, then #2, and finally #3. Turn in your answer sheet and the tie breaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Figures aren't necessarily drawn to scale. Angles are given in radians unless otherwise stated.

1. $\lim_{x\to 2}\left(\frac{4x^2-16}{x-2}\right)=16$. By the definition of a limit, there is a positive real number δ such that $\left|\frac{4x^2-16}{x-2}-16\right|<0.4$ if $0<|x-2|<\delta$. The largest valid value of δ is
   A. 0.02
   B. 0.05
   C. 0.1
   D. 0.2
   E. 0.5

2. $\lim_{x\to 0}\left(\frac{4x}{\sin(2x)}+\frac{5x}{\cos(3x)}\right)=$
   A. Undefined
   B. 1
   C. 2
   D. 4
   E. 9

3. Which of the following indicates the presence of a horizontal asymptote for the graph of $y = f(x)$?
   A. $\lim_{x\to 4} f(x)=3$
   B. $\lim_{x\to \infty} f(x)=3$
   C. $\lim_{x\to \infty} f(x)=\infty$
   D. $\lim_{x\to 3} f(x)=\infty$
   E. Each of the other answers is incorrect.

4. There is a stack of newspapers whose weight is given by $w(t)$ where t is time. A match is thrown in the stack and we notice that the fire is increasing in vigor at time $t = 2$. Which of the following must be true?
   A. $w'(t) > 0$ and $w''(t) > 0$
   B. $w'(t) > 0$ and $w''(t) < 0$
   C. $w'(t) < 0$ and $w''(t) > 0$
   D. $w'(t) < 0$ and $w''(t) < 0$
   E. One cannot determine the signs of these derivatives.

5. $\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} =$
   A. Does not exist
   B. 0
   C. $f(x)$
   D. $f'(x)$
   E. $f''(x)$

6. $\frac{d}{dx}\left(\ln\left(\sin(x)\right)\right) =$
   A. $\ln\left(\cos(x)\right)$
   B. $\cos\left(\ln(x)\right)$
   C. $\left(\frac{1}{x}\right)\cos(x)$
   D. $\sin\left(\frac{1}{x}\right)$
   E. $\cot(x)$

7. $\frac{d}{dx}\left(\frac{3}{x} + 4\sqrt{x} + 5x\right) =$
   A. $3\ln(x) + 2\sqrt{x} + 5$
   B. $-\frac{3}{x^2} + 2\sqrt{x} + 5$
   C. $-\frac{3}{x^2} + \frac{2}{\sqrt{x}} + 5$
   D. $3x + 4\sqrt{x} + 5$
   E. Each of the other answers is incorrect.

8. The depth of the water x feet from the end of a swimming pool is given by $h(x) = 3 + \frac{1}{80}x^2$ for $x \in [0,20]$. What is the average depth of the water on this interval to the nearest tenth of a foot?
   A. 3.7
   B. 4
   C. 4.3
   D. 4.7
   E. Each of the other answers is incorrect.

9. A region is bounded by the curves $x = 2$, $x = 4$, $y = x^4$, and $y = 4^x$. Compute the area of the region. Round your answer to two decimal places.
   A. 24.00
   B. 25.28
   C. 52.25
   D. 78.40
   E. Each of the other answers is incorrect.

For problems 10 and 11. Following is a table of velocities and times since midnight for a vehicle.

t hours	1	3	5	7	9	11	13	15	17
v miles/hour	25	45	53	55	60	62	61	53	42

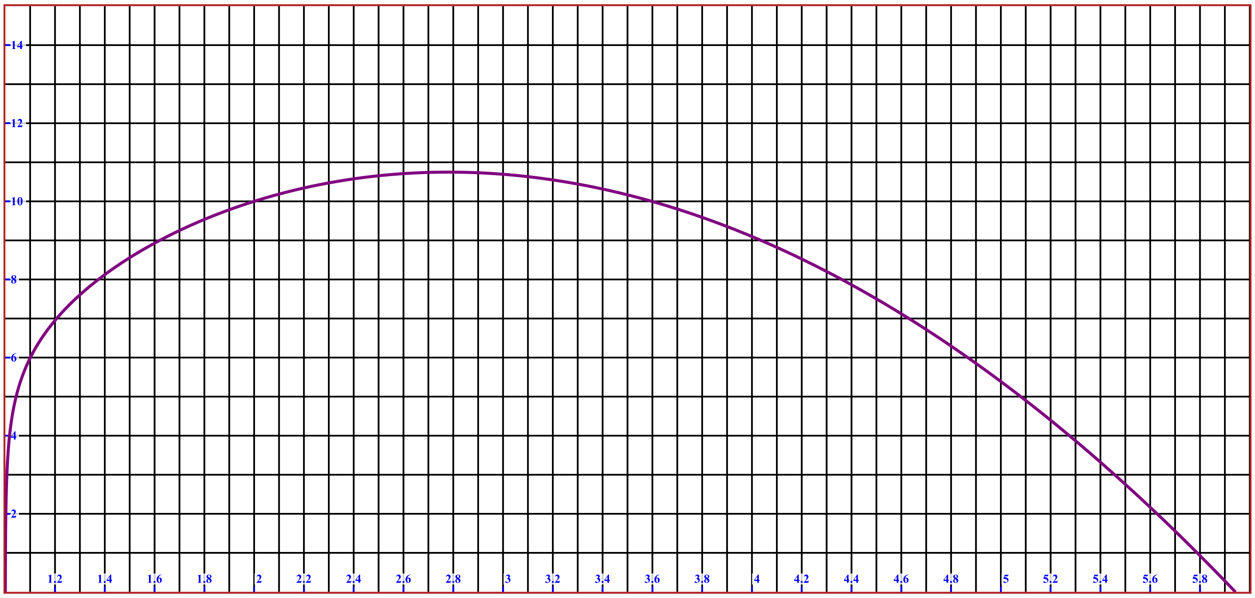
10. Give the best estimate of the instantaneous acceleration exactly at 7:00 am.
    - A. 1 mi/h^2
    - B. 1.75 mi/h^2
    - C. 2.5 mi/h^2
    - D. 7.86 mi/h^2
    - E. 55 mi/h^2

11. Use the midpoint rule with 4 intervals to approximate the total distance traveled from 1:00 am to 5:00 pm.
    - A. 860 mi
    - B. 828 mi
    - C. 430 mi
    - D. 203 mi
    - E. 180 mi

12. What is the equation of the tangent line to $f(x)=\frac{x^2+1}{x-1}$ at $x = 2$?
    - A. $y = 7 - x$
    - B. $y = x + 5$
    - C. $y = 5 - x$
    - D. $y = 2x + 5$
    - E. Each of the other answers is incorrect.

13. $\frac{d}{dx}\left(\sin^2(x)\right)=$
    - A. $2\sin(x)$
    - B. $\sin(2x)$
    - C. $\sin\left(x^2\right)$
    - D. $\cos^2(x)$
    - E. Each of the other answers is incorrect.

For problems 14 and 15. Following is the flow rate of a pollutant in a lake in liters/hour as a function of time in hours.



14. Estimate the total amount of the pollutant which entered the lake from time t = 2 to t = 5.
   A. 5 liters
   B. 10 liters
   C. 15 liters
   D. 20 liters
   E. 30 liters

15. How fast the flow rate changing at t = 4.4?
   A. Decreasing at 1.5 liters/hour/hour
   B. Decreasing at 2.5 liters/hour/hour
   C. Decreasing at 3.5 liters/hour/hour
   D. Increasing at 2.5 liters/hour/hour
   E. Increasing at 3.5 liters/hour/hour

16. $\frac{d}{dx}\left(\sec(x)\tan(x)\right) =$
   A. $2\sec^3(x) - \sec(x)$
   B. $\sec(x)$
   C. $\sec^3(x)\tan(x)$
   D. $\dfrac{\cos^2(x) - 2\sin^2(x)}{\cos^3(x)}$
   E. Each of the other answers is incorrect.

17. If $p(3) = 4$, $p'(3) = 0$ and $p''(3) = 2$ which of the following must be true about the graph of $s(x)$?
    A. The graph has a local maximum at (3, 4).
    B. The graph has a local minimum at (3, 4).
    C. The graph has an inflection point at (3, 4).
    D. There is a hole in the graph at (3, 4).
    E. Each of the other answers is incorrect.

18. $\frac{d}{dx}\left(e^{\sin\left(5x^3\right)}\right) =$
    A. $15x^2\cos\left(5x^3\right)e^{\sin\left(5x^3\right)}$
    B. $15x^2e^{\sin\left(5x^3\right)}$
    C. $15x^2e^{\cos\left(5x^3\right)}$
    D. $e^{\cos\left(15x^2\right)}$
    E. Each of the other answers is incorrect.

19. A container is in the shape of a square pyramid with the vertex at the bottom. Its base is 12 meters on each side, and its height is 4 meters. It is being filled with water at a rate of 9 m^3/min. How fast is the depth of the water growing when the depth is 2 meters? The volume of a pyramid is given by $V = \frac{1}{3}Bh$.
    A. 2 meters/minute
    B. 1 meter/minute
    C. 0.5 meters/minute
    D. 0.25 meters/minute
    E. Each of the other answers is incorrect.

20. There is a line going from the origin to a point on the graph of $y = x^2e^{-3x}$, $x \geq 0$. Of all such lines, what is the slope of the one with the largest slope?
    A. $\frac{1}{3e}$
    B. $\frac{1}{3}$
    C. $\frac{1}{2}$
    D. $\frac{1}{9e}$
    E. Each of the other answers is incorrect.

21. The region R bounded by the graphs of $y = x^{\frac{3}{2}}$, $y = 1$, and $x = 4$ is revolved around the y-axis to form a solid of revolution. The volume of this solid is given by the integral

A. $\pi \int_1^4 \left(x^{\frac{9}{4}} - 1 \right) dx$

B. $\pi \int_1^4 \left(x^3 - 1 \right) dx$

C. $\pi \int_1^{16} \left(x^{\frac{3}{2}} - 1 \right)^2 dx$

D. $\pi \int_1^{\sqrt[3]{16}} \left(1 - y^{\frac{4}{9}} \right) dy$

E. $\pi \int_1^8 \left(16 - y^{\frac{4}{3}} \right) dy$

22. Here is a table of values for a function $y = f(x)$:

x	2.9	2.99	2.999	2.9999	3	3.0001	3.001	3.01	3.1
$f(x)$	0.67872	0.67932	0.67984	0.67998	23	0.64002	0.64021	0.64235	0.64467

The values in this table suggest $\lim_{x \to 3} f(x) =$

A. 23
B. 0.68
C. 0.64
D. 0.66
E. The limit does not exist.

23. $\lim_{x \to \infty} \left(\frac{3x^7 + 7^{x-2}}{5x^7 - 3 \cdot 7^x} \right) =$

A. $-\frac{1}{3}$

B. $-\frac{1}{147}$

C. 0

D. $\frac{3}{5}$

E. ∞

24. $\lim_{x \to a}\left(\frac{\sin(x)-\sin(a)}{x-a}\right)=$

A. $-\cos(a)$

B. $\sin(1)$

C. 0

D. $\cos(a)$

E. None of the other answers is correct.

25. The function $g(x)$ has a derivative $g'(x)$ that is continuous over an interval $[a,b]$. The definite integral $\int_a^b g'(x)dx$ can be interpreted as

A. The net area between the graph of $y=g(x)$ and the x-axis between $x = a$ and $x = b$.

B. The average rate of change of $y=g(x)$ between $x = a$ and $x = b$.

C. The average rate of change of $y=g'(x)$ between $x = a$ and $x = b$.

D. The net change in the function $y=g(x)$ between $x = a$ and $x = b$.

E. The net change in the function $y=g'(x)$ between $x = a$ and $x = b$.

ACTM Regional Calculus Competition 2018

Tiebreaker Question 1

Name ____________________

School ____________________

Let $f(x)$ be a differentiable function and c be a constant real number. Let $g(x) = c \cdot f(x)$.

Complete the following statement: $g'(x) =$ _______________.

Prove this result.

ACTM Regional Calculus Competition 2018

Tiebreaker Question 2

Name ____________________

School ____________________

Consider the family of functions of the form $f(x) = \dfrac{ax}{bx + c}$, where *a*, *b*, and *c* are all non-zero real numbers.

Answer the following in terms of *a*, *b*, and *c*. Justify all answers.

a) Identify any discontinuities of the function. Determine whether the discontinuities are removable or non-removable.

b) Find $f'(x)$.

c) Find $f''(x)$

d) Determine the intervals where $f(x)$ is increasing or decreasing.

Tiebreaker Question 3

Name ____________________

School ____________________

The following table gives various values of a function and its derivatives.

x	$f(x)$	$f'(x)$	$f''(x)$
0	1	2	4
2	5	0	1
4	11	6	3

Furthermore, $f''(x)$ is continuous for all real numbers x.

Is it possible for the line $x = 3$ to be a vertical asymptote for the graph $y = f(x)$? Explain.

Solutions

1	C
2	C
3	B
4	D
5	E
6	E
7	C
8	D
9	B
10	B
11	A
12	A
13	B
14	E
15	C
16	A
17	B
18	A
19	D
20	A
21	E
22	E
23	B
24	D
25	D

ACTM Regional Calculus Competition 2018

Tiebreaker Question 1 *Solution*

Let $f(x)$ be a differentiable function and c be a constant real number. Let $g(x) = c\,f(x)$.

Complete the following statement: $g'(x)$ = **$\underline{c\,f'(x)}$**.

Prove this result.

Proof:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

Definition of Derivative

$$= \lim_{h\to 0} \frac{c\,f(x+h)-c\,f(x)}{h}$$

Definition of g(x)

$$= \lim_{h\to 0} \frac{c\left(f(x+h)-f(x)\right)}{h}$$

Distributive Property [$ca + cb = c(a + b)$]

$$= c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Limit Property:
$$\left[\lim_{h\to a}\left(c\,p(h)\right) = c\lim_{h\to a}\left(p(h)\right)\right]$$

$$= c\,f'(x)$$

Definition of Derivative

ACTM Regional Calculus Competition 2018

Tiebreaker Question 2 *Solution*

Consider the family of functions of the form $f(x) = \dfrac{ax}{bx+c}$, where *a*, *b*, and *c* are all non-zero real numbers. Answer the following in terms of *a*, *b*, and *c*. Justify all answers.

a) Identify any discontinuities of the function. Determine whether the discontinuities are removable or non-removable.

b) Find $f'(x)$.

c) Find $f''(x)$

d) Determine the intervals where $f(x)$ is increasing or decreasing.

a) Since *f* is a rational function, it is continuous throughout the domain. The only discontinuity is at the zero of the denominator, $x = -\dfrac{c}{b}$. At this value of *x*, the numerator will be $-\dfrac{ac}{b} \neq 0$ since *a* and *c* are both not zero. Thus, $\lim\limits_{x \to -\frac{b}{c}} f(x)$ does not exist, so *f* has a non-removable discontinuity.

b) Quotient Rule: $f'(x) = \dfrac{(bx+c)a - ax(b)}{(bx+c)^2} = \dfrac{ac}{(bx+c)^2}$ or

Product Rule $f'(x) = -ax(bx+c)^{-2}b + a(bx+c)^{-1} = (bx+c)^{-2}\left[-abx + a(bx+c)\right] = \dfrac{ac}{(bx+c)^2}$

c) $f''(x) = \dfrac{d}{dx}\left[ac(bx+c)^{-2}\right] = -2ac(bx+c)^{-3}b = -\dfrac{2abc}{(bx+c)^3}$

d) The denominator $(bx+c)^2$ of the derivative $f'(x)$ will be positive for all real values of *x* with $x \neq -\dfrac{c}{b}$. Thus the sign of $f'(x)$ will depend on the sign of the product *ac*.

- If *a* and *c* are both positive or both negative, then $f'(x)$ will be positive and thus $f(x)$ will be increasing on the intervals $\left(-\infty, -\dfrac{c}{b}\right)$ and $\left(-\dfrac{b}{c}, \infty\right)$.
- If *a* and *c* have different signs, then $f'(x)$ will be negative and thus $f(x)$ will be decreasing on the intervals $\left(-\infty, -\dfrac{c}{b}\right)$ and $\left(-\dfrac{b}{c}, \infty\right)$.

Tiebreaker Question 3 *Solution*
The following table gives various values of a function and its derivatives.

x	$f(x)$	$f'(x)$	$f''(x)$
0	1	2	4
2	5	0	1
4	11	6	3

Furthermore, $f''(x)$ is continuous for all real numbers x.

Is it possible for the line $x = 3$ to be a vertical asymptote for the graph $y = f(x)$? Explain.

No. The continuity of f'' implies that f' is differentiable, and thus continuous, for all x. Repeating the logic gives f is continuous for all x. Since f is continuous for all x, there can be no vertical asymptote.