Arkansas Council of Teachers of Mathematics
2019 Calculus Regional Exam

Work the multiple choice questions first, choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Then, attempt the tiebreaker questions at the end starting with Tie Breaker #1, then #2, and finally #3. Turn in your answer sheet and the tiebreaker pages when you are finished. You may keep the pages with the multiple-choice questions.

Figures aren't necessarily drawn to scale. Angles are given in radians unless otherwise stated.

1. Suppose $f(x)$ is a function such that $\frac{sinx}{x} \leq f(x) \leq 1$ for all x near 0. Use this fact to evaluate $\lim_{x\to 0} f(x)$.

   A. $\lim_{x\to 0} f(x) = 2$
   B. $\lim_{x\to 0} f(x) = 1$
   C. $\lim_{x\to 0} f(x) = 0$
   D. $\lim_{x\to 0} f(x)$ *does not exist*
   E. None of the above

2. Evaluate $\lim_{x\to\infty}\left(\frac{3x^2-2x}{\sqrt{4x^4+1}}\right) =$

   A. 0
   B. $\frac{3}{4}$
   C. $\frac{3}{2}$
   D. ∞
   E. The limit does not exist.

3. Let $f(x) = \frac{x^2}{3} + 2 \cdot \ln(3x^2) + e^3$. Find $f'(x)$.

   A. $f'(x) = \frac{2}{3}x + \frac{2}{3x^2} + 3e^2$
   B. $f'(x) = \frac{2}{3}x + \frac{2}{3x^2}$
   C. $f'(x) = \frac{2}{3}x + \frac{4}{x}$
   D. $f'(x) = \frac{2}{3}x + \frac{4}{x} + 6e$
   E. None of the above

4. Let $g(x) = 2x\tan(3x^2+4)$. Find $g'(x)$.

   A. $g'(x) = 12x^2\sec^2(3x^2+4) + 2\tan(3x^2+4)$
   B. $g'(x) = 12x\sec^2(3x^2+4) + 2\tan(6x)$
   C. $g'(x) = 2\sec^2(3x^2+4)\,(6x)$
   D. $g'(x) = 2x\sec^2(6x) + 2\tan(3x^2+4)$
   E. None of the above

5. Find the equation of the line tangent to the graph of $f(x) = 2e^x + 3x^2 + \cos^2 x$ when $x = 0$.

   A. $y = 2x + 3$
   B. $y = 2x - 6$
   C. $y = 3x + 2$
   D. $y = 3x - 6$
   E. None of the above

6. Suppose $2x^2y + 3x^4 + y = 3$. Find the slope of the tangent line at the point $(1,0)$.

   A. $-\frac{4}{5}$
   B. $\frac{4}{3}$
   C. -12
   D. -4
   E. None of the above

7. Suppose the area of a circle is increasing at a rate of k feet per second. How fast is the radius of the circle changing at the instant the radius is $3ft$? (Recall $A = \pi r^2$. Leave your answer in terms of π).

   A. $3k\pi$
   B. $\frac{\pi k}{2}$
   C. $\frac{6\pi}{k}$
   D. $\frac{k}{6\pi}$
   E. None of the above

8. Give the x values at which the absolute extrema occur (if they exist) for the function $f(x) = 2x^3 - 6x + 2$ on the interval $[2,4]$.

   A. absolute max at $x = 2$, absolute min at $x = 4$
   B. absolute min at $x = 2$, absolute max at $x = 4$
   C. absolute min at $x = 1$, absolute max at $x = 4$
   D. No absolute min, absolute max at $x = 4$
   E. None of the above

9. Of all rectangles having fixed perimeter p, determine the dimensions of the rectangle with the maximum area. Give your answers in terms of p.

   A. $2p \times 2p$
   B. $\frac{p}{4} \times \frac{p}{4}$
   C. $4p \times \frac{p}{4}$
   D. $\frac{p}{2} \times \frac{p}{2}$
   E. None of the above

10. Evaluate $\int_2^4 x^{\pi-1}\,dx$. Give an exact answer.

   A. $\frac{1}{\pi}(4^\pi - 2^\pi)$
   B. $\frac{1}{4}(\pi^2)$
   C. $\frac{1}{\pi-1}(2^{4\pi})$
   D. $\frac{1}{\pi}(2^\pi - 4^\pi)$
   E. None of the above

11. Suppose $\int_1^4 \left(\frac{a}{2}x - ax^2\right) dx = 5$, where a is a constant. Use this fact and properties of integrals to evaluate $\int_4^1 (ax - 2ax^2)\,dx$.

   A. 10
   B. -10
   C. 5
   D. -5
   E. None of the above

12. Find the indefinite integral: $\int \frac{2x+4}{3x^2+12x}dx$

A. $\frac{1}{3}\ln|3x^2+12x|+c$
B. $3\ln|3x^2+12x|+c$
C. $\frac{1}{3}(3\mathrm{x}^2+12x)^{-2}+c$
D. $3\ln|2x+4|+c$
E. None of the above

13. Find the indefinite integral: $\int \frac{x}{x+1}dx$

A. $\frac{x^2}{x^2+x}+c$
B. $x+\ln|x+1|+c$
C. $x-\ln|x+1|+c$
D. $\frac{1}{2}x^2+\ln|x+1|+c$
E. None of the above

14. Suppose $f(x)=x^2-ax$ where $a>0$. Determine the value of $b>0$ for which $\int_0^b f(x)dx=0$. (Note: Your answer will be in terms of the constant a).

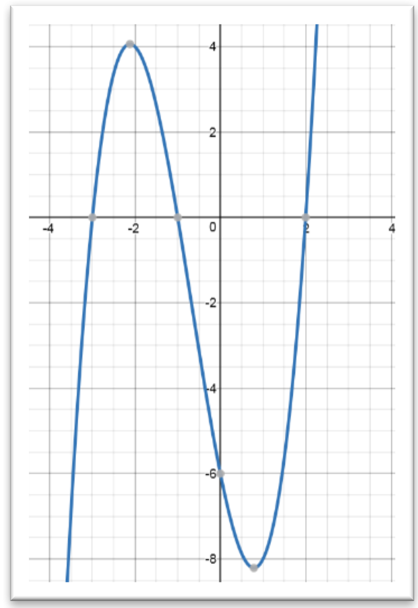
A. $b=3a$
B. $b=\frac{2}{3}a$
C. $b=2a+3$
D. $b=\frac{3}{2}a$
E. None of the above

15. Suppose $f(1)=3, f'(1)=2, g(1)=4$, and $g'(1)=0$. If $h(x)=\frac{2f(x)}{g(x)+1}$, find $h'(1)$.

A. $\frac{4}{5}$
B. $-\frac{2}{5}$
C. undefined
D. -4
E. None of the above

16. Determine if the Mean Value Theorem applies to the function $f(x) = \ln x + e$ on the interval $[1, e]$. If so, the Mean Value Theorem guarantees the existence of at least one x-value where the derivative has a certain value. If the theorem applies, find this/these x-value(s).

   A. $0, \frac{1}{2}$
   B. $2e + 1$
   C. $e - 1$
   D. The Mean Value Theorem does not apply.
   E. None of the above.



17. Given the graph of $f'(x)$ shown on the right, determine the x-value(s), if any, at which $f(x)$ has a relative maximum.

   A. $x = -3, 2$
   B. $x \approx -2.2$
   C. $x = -1$
   D. $x = -3, -1, 2$
   E. None of the above.

18. Use a Riemann sum to estimate the area bounded by the function $f(x) = x^2 + 1$, $x = 0$, $x = 8$, and the x-axis. Use a partition of 6 rectangles of equal width with right-hand end points.

   A. 138.37
   B. 178.67
   C. 223.70
   D. 260.00
   E. None of the above.

19. A car is headed west on a perfectly straight road with a speed limit of 55mph. At $t = 0$, the driver realizes that her speed is 80mph so she begins to slow down according to the acceleration function $a(t) = -1280(1 + 8t)^{-3}$ mph, where $t > 0$. How far does the driver travel between $t = 0$ and $t = 0.2$?

   A. 8.224 miles
   B. 6.154 miles
   C. 7.297 miles
   D. 5.003 miles
   E. None of the above.

20. Determine values of constants c and d so that the following function is continuous at $x = 2$.

$$f(x) = \begin{cases} 2x+3 & x<2 \\ d & x=2 \\ dx^2 - 7c & x>2 \end{cases}$$

A. $c = 3, d = 7$
B. $c = 3, d = 3$
C. $c = 7, d = \frac{1}{2}$
D. $d = 2, c = -3$
E. None of the above.

21. Let R be the region bounded by $y = x$, $y = 3x$, and $y = 4$. Find the volume of the solid generated when R is revolved about the y-axis.

A. $\frac{2}{3}\pi$
B. $\frac{256}{3}\pi$
C. $\frac{256}{27}\pi$
D. $\frac{512}{27}\pi$
E. None of the above.

22. Evaluate the following, where k is a constant.

$$\lim_{t\to 0} \frac{t}{\sqrt{kt+1} - 1}$$

A. ∞
B. k^2
C. $2k$
D. $\frac{2}{k}$
E. None of the above.

23. Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$ and $f''(c) < 0$. Which of the following are true?

A. f has a relative maximum at c.
B. f is undefined at c
C. f has a relative minimum at c
D. $\big(c, f(c)\big)$ is a point of inflection
E. None of the above.

24. Determine the equation of the line that represents the linear approximation of $f(x) = \sin x + \ln(1+x)$ at $x = 0$.

    A. $L(x) = 2x$
    B. $L(x) = x + 2$
    C. $L(x) = \frac{1}{2}x$
    D. $L(x) = \frac{1}{2}x + 2$
    E. None of the above

25. Evaluate the following limit, where a and b are constants and $b \neq 0$.

$$\lim_{x\to 0}\left(\frac{3\sin(ax)}{2bx} + \frac{1-\cos x}{x}\right)$$

    A. 0
    B. $\frac{3a}{2b}$
    C. $\frac{a-1}{b}$
    D. $\frac{3}{2}a + 2$
    E. None of the above

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Tie Breaker #1

Name: ___________________

School: __________________

Suppose $(x) = \begin{cases} \frac{9-x}{\sqrt{x}-3} & x < 9 \\ b+3 & x \geq 9 \end{cases}$.

Determine the value of the constant b for which $\lim_{x \to 9} f(x)$ exists. Show all of your work.

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Tie Breaker #2

Name: ____________________

School: __________________

Of all boxes with a square base and a volume of $k\ ft^3$, which one has the minimum surface area? Give the dimensions in terms of the constant k. Show all work.

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Tie Breaker #3

Name: ___________________

School: __________________

Two cylindrical tanks are filled with water at the same rate. The larger tank has a radius of 8 ft while the smaller tank has a radius of 5 ft. Suppose the water level in the smaller tank is rising at a rate of 0.5 ft/min. How fast is the water level rising in the larger tank?

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Answer Key

1. B
2. C
3. C
4. A
5. A
6. D
7. D
8. B
9. B
10. A
11. B
12. E
13. C
14. D
15. A
16. C
17. C
18. C
19. B
20. A
21. D
22. D
23. A
24. A
25. B

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Tie Breaker 1 Solution

Suppose $(x) = \begin{cases} \frac{9-x}{\sqrt{x}-3} & x < 9 \\ b+3 & x \geq 9 \end{cases}$.

Determine the value of the constant b for which $\lim_{x \to 9} f(x)$ exists. Show all of your work.

$$\lim_{x \to 9^-} f(x) = \lim_{x \to 9^-} \frac{9-x}{\sqrt{x}-3}$$

$$= \lim_{x \to 9^-} \frac{(9-x)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x \to 9^-} \frac{(9-x)(\sqrt{x}+3)}{x-9}$$

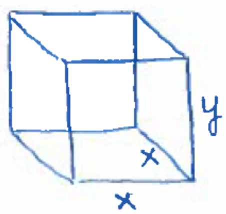
$$= \lim_{x \to 9^-} \frac{-\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{x-9}} = \lim_{x \to 9^-} -(\sqrt{x}+3) = -6$$

$$\lim_{x \to 9^+} f(x) = \lim_{x \to 9^+} (b+3) = -6$$

$$b + 3 = -6$$
$$b = -9$$

Tie Breaker 2 Solution

Of all boxes with a square base and a volume of $k\ ft^3$, which one has the minimum surface area? Give the dimensions in terms of the constant k. Show all work.



$V = x^2 y = k\ ft^3$

$SA = 2x^2 + 4xy$

$x^2 y = k$

$y = \frac{k}{x^2}$

$\Rightarrow SA(x) = 2x^2 + 4x\left(\frac{k}{x^2}\right)$

$SA(x) = 2x^2 + 4kx^{-1}$

$SA'(x) = 4x - 4kx^{-2}$

$4x - \frac{4k}{x^2} = 0$

$4x^3 - 4k = 0$

$4x^3 = 4k$

$x^3 = k$

$x = \sqrt[3]{k}$

$SA''(x) = 4 + 8kx^{-3}$

$SA''(k^{1/3}) = 4 + 8k(k^{1/3})^{-3}$

$= 4 + 8k(k^{-1})$

$= 4 + 8 = 12 > 0$

So, by the second deriv. test, there is a rel. max at $x = k^{1/3}$

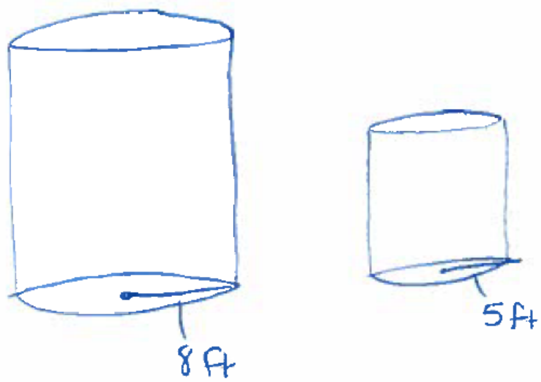
$y = \frac{k}{(\sqrt[3]{k})^2} = \frac{k}{k^{2/3}} = k^{1/3}$

Dimensions:

$k^{1/3} \times k^{1/3} \times k^{1/3}$

Tie Breaker 3 Solution

Two cylindrical tanks are filled with water at the same rate. The larger tank has a radius of 8 ft while the smaller tank has a radius of 5 ft. Suppose the water level in the smaller tank is rising at a rate of 0.5 ft/min. How fast is the water level rising in the larger tank?



$$V = \pi r^2 h$$

$$V_L = \pi (8)^2 h_L \qquad V_S = \pi (5)^2 h_S = 25\pi h_S$$

$$\frac{dV_S}{dt} = 25\pi \frac{dh_S}{dt}$$

$$\frac{dV_S}{dt} = 25\pi (.5) = \frac{25}{2}\pi$$

$$\frac{dV_L}{dt} = 64\pi \frac{dh_L}{dt} = \frac{dV_S}{dt} = \frac{25}{2}\pi$$

$$64\pi \frac{dh_L}{dt} = \frac{25}{2}\pi$$

$$\frac{dh_L}{dt} = \frac{25}{128} \text{ ft/min}$$

$$\approx .195 \text{ ft/min}$$