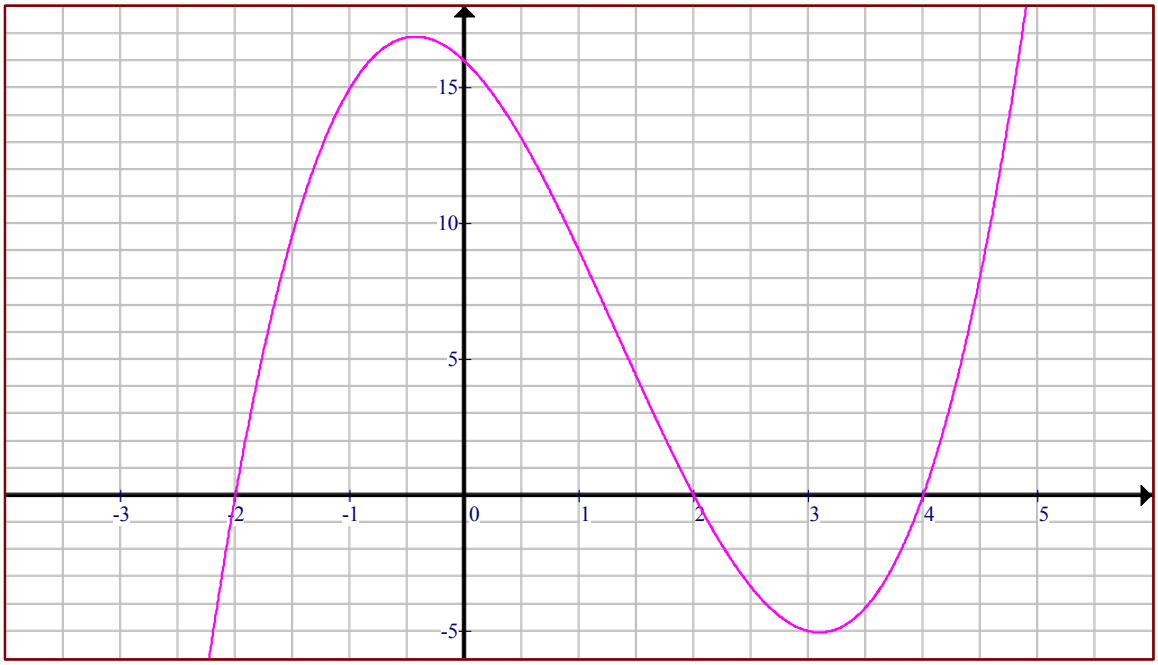
2016 ACTM Regional Algebra II Competition

Begin by removing the three tie breaker sheets at the end of the exam and writing your name on all three pages. Work the multiple choice questions first choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Make sure you attempt the tie-breaker questions at the end of the test starting with tie breaker 1, then 2, and then 3 if you have time. Turn in your answer sheet and the tie breaker pages when you are finished. You may keep the pages with the multiple choice questions.

1. $g(x) = 2x^2 - 3x + 4$. *f(x)* is given by the graph:



x	*h(x)*
-2	1
-1	0
0	2
1	-1
2	1

$h \quad f \quad g(0) =$

A. 409
B. 4
C. -1
D. 2
E. Each of the other answers is incorrect.

2. The ***sum*** of the solutions to $6x^3 + 7x^2 - x - 2 = 0$ is
   A. $\frac{1}{6}$
   B. $\frac{1}{3}$
   C. $-\frac{7}{6}$
   D. 2
   E. Each of the other answers is incorrect.

3. How many solutions does $2\sin(3x) = 1$ have?
   A. 1
   B. 2
   C. 3
   D. 6
   E. Each of the other answers is incorrect.

4. Solve the following system of equations:

$$\begin{aligned} 2x + 5y - 6z &= -15 \\ -1x + 3y + 4z &= 4 \\ 3x + y + z &= 4 \end{aligned}$$

   The ***sum*** of the three coordinates of the solution is:
   A. 0
   B. 1
   C. 2
   D. 3
   E. Each of the other answers is incorrect.

5. What are the vertical asymptote(s) of $f(x) = \dfrac{3x^2 + 6x - 24}{2x^2 + 2x - 12}$?
   A. $x = 1.5$
   B. $x = -3$
   C. $x = 2$ and $x = -3$
   D. $x = 2$
   E. Each of the other answers is incorrect.

6. Divide $(2x^4 + x^3 + 6x^2 - 7x - 7)$ by $(2x + 1)$. What is the remainder?
   A. -5
   B. 2
   C. -2
   D. 0
   E. Each of the other answers is incorrect.

7. The following graphs temperature in ^{o}F as a function of time in hours since noon at a certain location. What is the average rate of change in the temperature over the time period graphed?

A. 4^{o}F
B. -4^{o}F/hour
C. -0.5^{o}F/hour
D. -2^{o}F/hour
E. Each of the other answers is incorrect.

8. $2000 is deposited in a 4-year certificate of deposit which pays an annual rate of 1.5% compounded monthly. What is the value of the CD at maturity?
A. $2,107.74
B. $2,120.00
C. $2,122.73
D. $2,123.59
E. $2,431.01

9. $\ln\left(\frac{x}{y}\right) =$
A. $\ln(x) - \ln(y)$
B. $\ln(x - y)$
C. $\dfrac{\ln(x)}{\ln(y)}$
D. $\log_y(x)$
E. Each of the other answers is incorrect.

10. Which of the following is the correct next step used to simplify the following expression so that it only uses reduced radicals and rationalized denominators:

$$\frac{5x^2y}{\sqrt[5]{16x^5y^3z^4}}$$

A. $\dfrac{5x^2y}{\sqrt[5]{16x^5y^3z^4}} = \dfrac{5x^2y}{\sqrt[5]{16x^5y^3z^4}} \cdot \dfrac{\sqrt[5]{2y^2z}}{\sqrt[5]{2y^2z}}$

B. $\dfrac{5x^2y}{\sqrt[5]{16x^5y^3z^4}} = \dfrac{5x^2y}{\sqrt[5]{16x^5y^3z^4}} \cdot \dfrac{\sqrt[5]{16x^5y^3z^4}}{\sqrt[5]{16x^5y^3z^4}}$

C. $\dfrac{5x^2y}{\sqrt[5]{16x^5y^3z^4}} = \dfrac{5x^2y}{\sqrt[5]{16x^5y^3z^4}}$

D. $\dfrac{5x^2y}{\sqrt[5]{16x^5y^3z^4}} = \dfrac{5}{\sqrt[5]{16x^3y^2z^4}}$

E. Each of the other answers is incorrect.

11. $f(x) = x^{\frac{p}{q}}$ where the exponent is a reduced rational number with p odd and q even. The graph of the function is in
    A. Quadrant I only
    B. Quadrants I and II
    C. Quadrants I and III
    D. Quadrants II and IV
    E. Not enough information is given to determine which quadrants have portions of the graph.

12. What is the next step in solving the equation $x^2 + 6x = 2$ by completing the square?
    A. $x^2 + 6x - 9 = 2 - 9$
    B. $\dfrac{x^2 + 6x}{6} = \dfrac{2}{6}$
    C. $x^2 + 6x - 2 = 2 - 2$
    D. $x^2 + 6x + 9 = 2 + 9$
    E. Each of the other answers is incorrect.

13. A population of fish was started with 400 fish. The population has been growing at a relative rate of 6% per year. Assuming that this growth pattern continues how long will it be before the population triples from its current size?

A. $\frac{\ln(3)}{\ln(1.06)}$ years

B. $\frac{\ln(1.06)}{\ln(3)}$ years

C. $\frac{\ln(3)}{\ln(0.06)}$ years

D. $\ln(3)\ln(0.06)$ years

E. Not enough information is given to determine an answer or the other answers are all incorrect.

14. If $y = p(x)$ is a third degree polynomial function with real coefficients, then which of the following statements are always true for any such function?

I. The range of p is the set of all real numbers.
II. The graph of p touches the x-axis in at least three different places.
III. The graph of p does not have any vertical or horizontal asymptotes.
IV. The graph of p touches the x-axis in less than four different places.

A. I, II, III, and IV
B. I, III, and IV
C. III and IV
D. I and IV
E. Each of the other answers is incorrect.

15. The graph of a polynomial function with degree n has at most how many local extrema?
A. n
B. $n - 1$
C. $n - 2$
D. $n + 1$
E. Each of the other answers is incorrect.

16. Describe the graph of $3x^2 - 4x - 3y^2 + 2y - 7 = 0$.
A. Circle
B. Ellipse
C. Parabola
D. Hyperbola
E. Each of the other answers is incorrect.

17. Suppose that you roll a pair of fair standard six-sided dice and record the total number of dots on the upward faces (i.e. the sum of the two dice). What is the probability that this sum will be a 7?
    A. $\frac{1}{11}$
    B. $\frac{1}{12}$
    C. $\frac{1}{6}$
    D. $\frac{1}{7}$
    E. $\frac{1}{4}$

18. John scores on free throws 60% of the time. Whenever John is fouled while shooting, John is awarded two free throws. What is the average number of points John will make when he is awarded two free throws? Each made free throw is worth one point.
    A. 0.72
    B. 0.98
    C. 1
    D. 1.2
    E. 2

19. If $\cos(\alpha) = -\frac{5}{13}$ and $\cot(\alpha) > 0$, then $\sin(\alpha) =$
    A. $-\frac{12}{13}$
    B. $\frac{5}{13}$
    C. $\frac{12}{13}$
    D. $-\frac{13}{5}$
    E. Each of the other answers is incorrect.

20. The probability that a random person has a particular disease is 5%. The probability that a test for the disease is accurate if the person has the disease is 95%. What is the probability that a person chosen at random will have the disease and test negative for it?
    A. 4.75%
    B. 0.25%
    C. 90.25%
    D. 5.00%
    E. Each of the other answers is incorrect.

21. If $r(x)$ is a rational function then which of the following cannot be true?
    A. The graph of $r(x)$ is continuous.
    B. The graph of $r(x)$ has a horizontal asymptote on the left and right.
    C. The graph of $r(x)$ has a vertical asymptote.
    D. The graph of $r(x)$ has a hole in the graph.
    E. The graph of $r(x)$ has a horizontal asymptote on the left but not on the right.

22. We are given the system of equations:

$$2x + 3y = 7$$
$$5x - 6y = 9$$

Solving the first equation for x and substituting in the second equation results in

A. $5\left(\frac{7-3y}{2}\right) - 6y = 9$

B. $5x - 6\left(\frac{7-3y}{2}\right) = 9$

C. $5\left(\frac{7-3y}{2}\right) - 6\left(\frac{7-2x}{3}\right) = 9$

D. $2x + 3\left(\frac{7-2x}{3}\right) = 7$

E. $2\left(\frac{7-2x}{3}\right) + 3y = 7$

23. How does the graph of $y = \ln(2x) + 5$ compare to the graph of $y = \ln(x)$?
    A. It is shifted to the left 2 units and expanded vertically away from the x-axis by a factor of 5.
    B. It is shifted up 5 units and expanded horizontally away from the y-axis by a factor of 2.
    C. It is shifted up 5 units and compressed horizontally toward the y-axis by a factor of ½.
    D. It is shifted down 5 units and compressed horizontally toward the y-axis by a factor of ½.
    E. Each of the other answers is incorrect.

24. Suppose a process is to add three to an input, multiply the result by 2, and cube this result. What describes the inverse process?
    A. Subtract 3 from an input, divide the result by 2, and cube this result
    B. Subtract 3 from an input, divide this result by 2, and take the cube root of this result
    C. Take an output and add three, multiply the result by 2, and cube this result
    D. Take the cube root of an input, divide this result by 2, and subtract 3 from this result
    E. Each of the other answers is incorrect.

25. Solve the equation $3y^2D - 5 + x^2D + 2xy = 0$ for D.

A. $D = \frac{3}{x+3y}$

B. $D = \frac{5-2xy-x^2D}{3y^2}$

C. $D = \frac{5-2xy}{x^2+3y^2}$

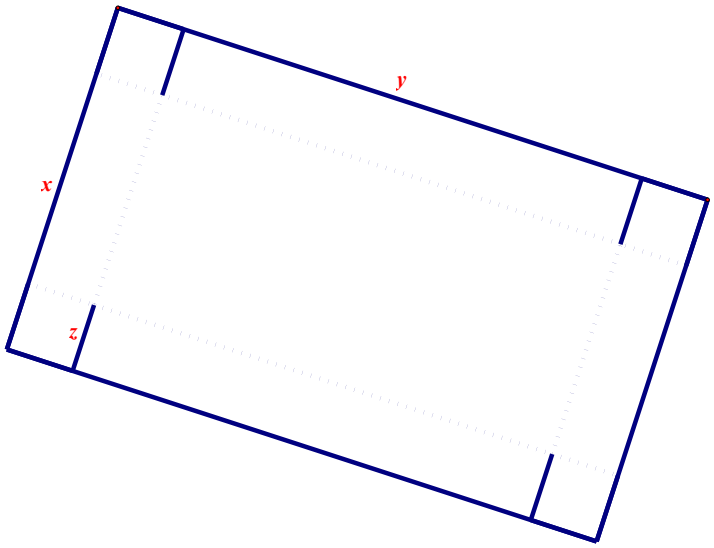
D. $D = \frac{5-2xy}{3x^2y^2}$

E. Each of the other answers is incorrect.

ACTM Regional Algebra II Competition 2016
Tie Breaker 1

Name _____

Suppose that you are given a rectangular piece of cardboard which is $x = 5$ cm wide and $y = 10$ cm long. Suppose that we go in a distance z from each vertex and make a cut of length z perpendicular to the side (solid lines in the diagram below). We then fold on the dotted lines in the diagram and tape the square tabs that we form at each corner to make a box without a top in the shape of a right rectangular prism. What should the value of z be in order to maximize the volume of such a box? What is this maximum volume? Give your answers rounded to four decimal places.



ACTM Regional Algebra II Competition 2016
Tie Breaker 2

Name _____

Create a formula for a rational function which has all of the following characteristics

- A domain of all real numbers except 3
- A hole in the graph at (3, 4)
- A horizontal asymptote of $y = 0$ on both sides of the graph

ACTM Regional Algebra II Competition 2016
Tie Breaker 3

Name _____

John works 250 days a year. Each day on his way to work, John has been stopping at a coffee shop and purchasing a cappuccino for $3.50. However, he makes a new year's resolution to stop purchasing the daily cappuccino and start saving the money instead. Thus, on January 1, he places an empty jar on his dresser, and each work day John puts $3.50 in the jar. At the end of one year, he empties the jar and invests the money in an account paying 4% annual interest (funds must be in the account for a year to earn interest). He continues adding $3.50 each working day to the jar on the dresser, and investing the total from the jar into the account at the end of each year. After three years (including the deposit from the jar at the end of the third year), what will be the account balance?

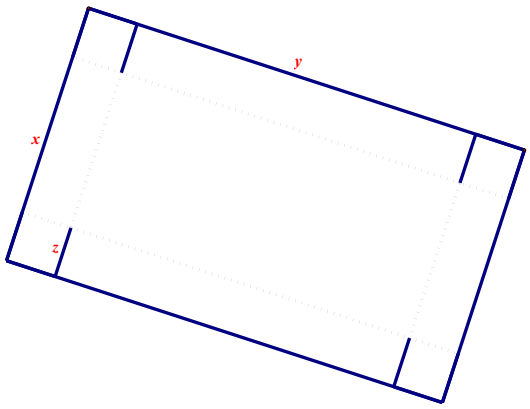
ANSWERS:

1. D
2. C
3. E (infinitely many)
4. C
5. B
6. C
7. B
8. D
9. A
10. A
11. A
12. D
13. A
14. B
15. B
16. D
17. C
18. D
19. A
20. B
21. E
22. A
23. C
24. D
25. C

ACTM Regional Algebra II Competition 2016
Tie Breaker 1

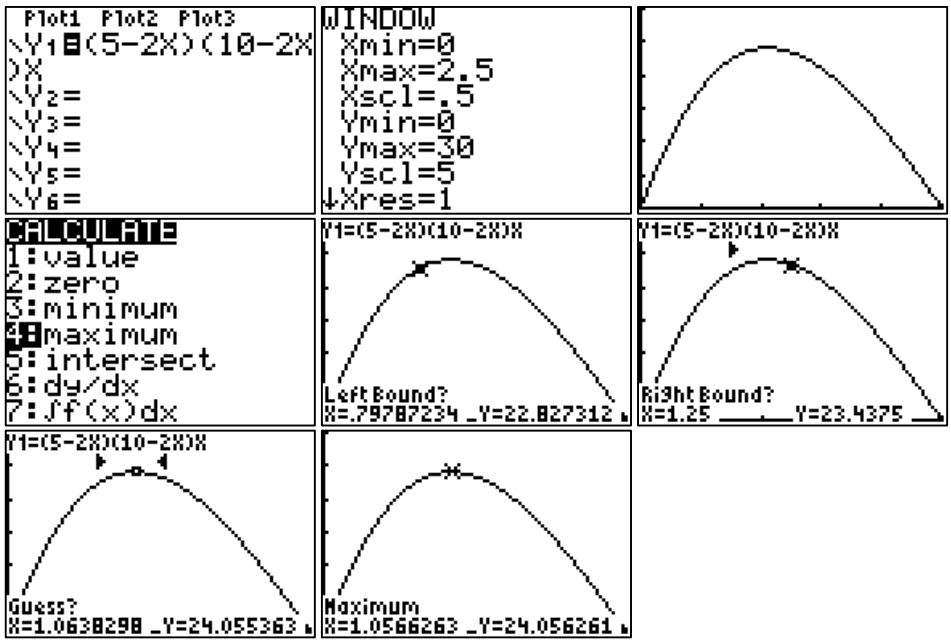
Name _____

Suppose that you are given a rectangular piece of cardboard which is $x = 5$ cm wide and $y = 10$ cm long. Suppose that we go in a distance z from each vertex and make a cut of length z perpendicular to the side (solid lines in the diagram below). We then fold on the dotted lines in the diagram and tape the square tabs that we form at each corner to make a box without a top in the shape of a right rectangular prism. What should the value of z be in order to maximize the volume of such a box? What is this maximum volume? Give your answers rounded to four decimal places.



Note that after the cuts and folds are made the dimensions of the box will be
width = $x - 2z = 5 - 2z$
length = $y - 2z = 10 - 2z$
height = z
Therefore its volume = V = (width)(length)(height)
so $V = (5 - 2z)(10 - 2z)\, z$
Note that width must be positive so $5 - 2z > 0$
which implies $5 > 2z$ so $z < 2.5$.
And the height is positive so $z > 0$.
So we see that $0 < z < 2.5$.

Graph the volume function on a graphing calculator using this restriction for the domain to get window settings. Then use the capabilities of the calculator to approximate the coordinates of the local maximum via CALC Maximum.



When z is approximately 1.0566 cm the volume will be maximized at approximately 24.0563 cm^3.

ACTM Regional Algebra II Competition 2016
Tie Breaker 2

Name _____

Create a formula for a rational function which has all of the following characteristics

- A domain of all real numbers except 3
- A hole in the graph at (3, 4)
- A horizontal asymptote of $y = 0$ on both sides of the graph

- Since the domain is all real numbers except 3 the only factor of the denominator which can equal zero must be $(x - 3)$.

- There is a hole in the graph at (3, 4) so $(x - 3)$ must also be a factor of the numerator. Furthermore, reducing the function and substituting in $x = 3$ should result in an output of 4.

- Since the function has a horizontal asymptote of 0 on both sides the degree of the denominator must be larger than the degree of the numerator. This forces the denominator to have a quadratic factor which is never zero for any real number input. There are infinitely many such quadratic factors with $(x^2 + 1)$ perhaps being the simplest.

So there are infinitely many such functions one of which is

$$f(x) = \frac{40(x-3)}{(x^2+1)(x-3)}.$$

ACTM Regional Algebra II Competition 2016
Tie Breaker 3

Name _____

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Answer: $2731.40
Amount collected annually in jar: 250*$3.50 = $875.

Method 1: Iterative approach:
End of Year 1: $875
End of Year 2: Balance + Interest + Deposit = 875 + .04(875) + 875 = 875+35+875 = $1785
End of Year 3: Balance + Interest + Deposit = 1785 + .04(1785) + 875 = 1785+71.40+875 = $2731.40

Method 2: Iterative using 1+r:
End of Year 1: $875
End of Year 2: 875(1.04) + 875 = 910+875 = $1785
End of Year 3: 1785(1.04) + 875 = 1856.40+875 = $2731.40

Method 3: Expanded iterative:
((875*1.04+875)*1.04)+875 = $2731.40

Method 4: Geometric Series:
$875 + 875*1.04 + 875*1.04^2 = 875 + 910 + 946.40 = \$2731.40$
Or

$$875 + 875*1.04 + 875*1.04^2 = \frac{875\left(1.04^3 - 1\right)}{1.04 - 1} = \$2731.40$$