2009 ACTM State Math Contest
Calculus

1. Let [x] be the largest integer less than or equal to x. Find $\lim_{x\to 2.5}[x+3]$

   a. 5 b. 6 c. 3 d. does not exists e. none of the above

2. If f(x) = $\frac{x^2+x-12}{x^2-5x+6}$ $(x\neq 3)$ and f is continuous at x =3, then f(3) is equal to

   a. -2 b. -11 c. 7 d. does not exist e. none of the above

3. $\lim_{x\to 0}\frac{e^{\tan x}-e^x}{\tan x+x}$ is equal to

   a. -1 b. 0 c. 1 d. does not exist e. none of the above

4. Evaluate $\lim_{x\to 0}\frac{2^x - 3^x}{x}$

   a. 0 b. 1 c. ln (2/3) d. does not exist e. none of the above

5. $\lim_{n\to\infty}\left(\frac{1+2^{199}+3^{199}+\ldots+n^{199}}{n^{200}}\right)$ is equal to :

   a. 1/201 b. 1/200 c. 1/199 d. 1/198 e. none of the above

6. Compute $\frac{dy}{dx}$ where y = cos (2x).

   a. *sin (2x)* b. –sin (2x) c. 2 sin (2x) d. – 2sin(2x) e. none of the above

7. If $y = x^3 + 2x$ and $\frac{dx}{dt}=5$, find $\frac{dy}{dt}$ when x = 2.

   a. 14 b. 62 c. 67 d. 135 e. none of the above

8. Let $\frac{1}{x+y}(1+\frac{dy}{dx}) - 2y - 2x\frac{dy}{dx} = 0;\ y(0) = 2.$ Find $y'(0)$.

   a. 1/7 b. 1/6 c. 6 d. 7. e. none of the above

9. Let y be a function of x which satisfies ln (x+y) – 2xy = 0 and y(0) = 1. Find $y'(0)$.

a. -1 b. -1/2 c. 0 d. 1 e. none of the above

10. Find equation of the tangent line to y = x^2 + 1 which is parallel to the line y = -4x.

a. 4x + y – 3 = 0
b. 4x + y + 3 = 0
c. 4x-y-13 = 0
d. 4x –y + 13 = 0
e. None of the above

11. Find equation of the tangent line to $y = \sqrt{5 + x^2}$ at x = 2.

a.2x + 3y + 5 = 0 b. 2x+3y -5 = 0 c. 2x-3y-5 = 0 d. 2x-3y+5=0 e. none of the above

12. Find the nth derivative of the function f(x) = e^{3x}

a. $3^n e^{3x}$ b. $3^{n-1}e^{3x}$ c. $3^n e^x$ d. e^{3nx} e. None of the above

13. If f(x) = $\frac{x^2}{x+5}$, then f''(5) is equal to

a. 1/20 b. 5/2 c. 2 d. undefined e. none of the above

14. Let f(x) = $\frac{1}{x}\int_3^x (2t-3f'(t))dt$. Find $f'(3)$.

a. -1 b. -1/3 c. 1/3 d. 1 e. none of the above

15. If $y = e^{-x^2}\int_0^x e^{t^2}\,dt$ then $\frac{dy}{dx}$ + 2xy is equal to

a. -1 b. 0 c. 1 d. 2 e. None of the above

16. Evaluate $\int_{-1}^{1} f(x)\,dx$ where f(x) = $\begin{cases} 1+2x & if \quad x \geq 0 \\ x^2 & if\, x<0 \end{cases}$

a. 1/3 b. 4/3 c. 7/3 d. 8/3 e. None of the above.

17. $\int_{-1}^{1} (ax^3 + bx)\, dx = 0$ for

a. Only for a > 0 and b > 0
b. Only for a < 0 and b < 0
c. Only for a > 0 and b < 0
d. Only for a < 0 and b > 0
e. None of the above

18. Evaluate $\int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx$

a. ½ ln(cos x + sin x) + c
b. ½ ln (1 + sin 2x) + c
c. ½ ln (cos x – sin x) + c
d. ½ ln(1 – sin 2x) + c
e. None of the above

19. Suppose f is an integrable function on [0,1]. Evaluate $\int_{0}^{1} \frac{f(x)}{f(x) + f(1-x)}\, dx$

a. $\frac{f(1) + 1}{f(1) - 1}$
b. $\frac{f(1)}{f(1) + 1}$
c. $\frac{f(1)}{f(1) - 1}$
d. $\frac{f(1) - 1}{f(1) + 1}$
e. None of the above

20. The maximum value of xt subject to x + t = 9 is:

a. 20.00 b. 20.25 c. 20.5 d. 20. 75 e. none of the above

21. Find two positive numbers whose product is 100 and whose sum is minimum.

a. 5 and 20 b. 4 and 25 c. 2.5 and 40 d. 2 and 50 e. none of the above

22. Find all numbers c for the function $f(x) = \frac{x}{x+2}$ on [1,4] that satisfy the conclusion of the Mean Value Theorem.

a. 2.5 b. $-2+2\sqrt{3}$ c. $-2+\sqrt{6}$ d. $-2+3\sqrt{2}$ e. none of the above

23. Let f: R $\rightarrow$ R b defined by f(x) = x –[x], where [x] is the greatest integer not exceeding x. Then the set of discontinuities of f is equation.

a. Set of all real numbers
b. set of all rational numbers
c. set of all integers
d. set of all irrational numbers
e. none of the above

24. Two sides of a triangle are 4 meters and 5 meters in length and the angle between them is increasing at a rate of 0.06 radians per second. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$.

a. 1.25 sq meters/sec
b. 0.625 square meters/sec
c. 0.18 square meters/sec
d. 0.15 square meters/second
e. none of the above

25. The maximum value of $f(x) = 2x^3 - 21x^2 + 36x + 20$ on the interval [0,2] is:

a. 20 b. 24 c. 37 d. 39 e. None of the above

Tie Breaker #1

Evaluate $\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}$. (Show your work.)

Tie Breaker #2

In the first quadrant, find the point on the parabola $y = 1 - x^2$ at which the tangent forms the triangle with the smallest area.

Tie Breaker #3

Show that $\frac{1}{17} \leq \int_1^2 \frac{1}{1+x^4}\, dx \leq \frac{7}{24}$.

2009 ACTM State Math Contest

Calculus

Answer Key

1. a
2. c
3. b
4. c
5. b
6. d
7. e
8. d
9. d
10. b
11. d
12. a
13. a
14. d
15. c
16. c
17. e
18. b
19. e
20. b
21. e
22. d
23. c
24. e
25. c

Tie Breaker #1

Let -1/2 < x < ½. Then, for x $\neq$ 0,

$$\frac{|2x-1|-|2x+1|}{x} = \frac{-(2x-1)\ -(2x+1)}{x}$$

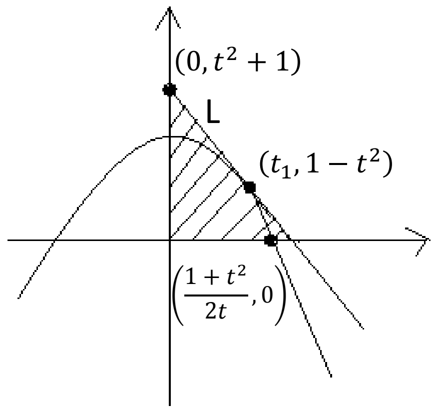
$$= \frac{-2x+1-2x-1}{x}$$

$$= \frac{-4x}{x} = -4$$

Therefore $\displaystyle\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}$ = -4 .

Tie Breaker # 2

Let $(t, 1 - t^2)$ is the desired point on the parabola in the first quadrant . See the figure. The slope of the tangent line L at $(t, 1 - t^2)$ is equal to $\frac{dy}{dx}$ $at\ x = t$, which is -2t. An equation of the tangent line to the parabola at the point $(t, 1 - t^2)$ is $y - 1 + t^2 = -2t(x-t)$.



Therefore, the x- & y- intercepts, respectively, are $\left(\frac{1+t^2}{2t}, 0\right)$ and $(0, t^2 + 1)$. Hence the area, A(t) of the triangle (as shown in the figure) is:

$$A(t) = \frac{(1+t^2)^2}{4t} = \frac{1}{4t} + \frac{t}{2} + \frac{t^3}{4}$$

Now set A'(t) = 0. This implies

$$\frac{-1}{4t^2} + \frac{1}{2} + \frac{3t^2}{4} = 0$$

$$\Rightarrow \quad 3t^4 + 2t^2 - 1 = 0$$

$$\Rightarrow \quad (3t^2 - 1)(t^2 + 1) = 0 \Rightarrow \quad t^2 = \frac{1}{3} \Rightarrow \quad t = \frac{\sqrt{3}}{3}$$

Therefore the desired point is $\left(\frac{\sqrt{3}}{3}, \frac{2}{3}\right)$.

Tie Breaker # 3:

On the interval [1,2], $1 + x^4 \leq 17$.

Hence $\int_1^2 \frac{1}{1+x^4}\, dx \geq \int_1^2 \frac{1}{17}\, dx = \frac{1}{17}$.

On the other hand,

$$\int_1^2 \frac{1}{1+x^4}\, dx \leq \int_1^2 \frac{1}{x^4} dx = \frac{7}{24}.$$

Thus, $\frac{1}{17} \leq \int_1^2 \frac{1}{1+x^4}\, dx \leq \frac{7}{24}.$