2009 Regional Math Contest
Calculus

Select the best answer for each of the following questions and mark it on the answer sheet provided. Be sure to read all the answer choices before making your selection. When you are finished with the multiple-choice questions, please attempt the tiebreaker questions.

1. Find $\lim_{x\to 1+} \dfrac{\sqrt{x^2-1}+\sqrt{x-1}}{\sqrt{x^2-1}}$.

   a. 1 b. $\dfrac{2+\sqrt{2}}{2}$ c. $\dfrac{1+\sqrt{2}}{2}$ d. Does not exist e. None of the above

2. Compute : $\lim_{x\to\infty} \dfrac{2x+5\sin x}{3x+8\cos x}$

   a. 7/11 b. 23/12 c. 2/3 d. 5/8 e. ∞

3. Compute $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{i}{n})(\frac{2}{n})$

   a. 1 b. 3/2 c. 3 d. 4 e. ∞

4. Find the value of a such that $\lim_{x\to 0} \dfrac{\tan(ax+\pi/4)-1}{x} = 4.$

   a. 1 b. 2 c. $2\sqrt{2}$ d. 16/3 e. None of the above

5. Compute the limit of the following Riemann sum: $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{i}{n^2+i^2}$

   a. $\dfrac{1}{4}-\ln 2$ b. $\dfrac{1}{3}-\ln 2$ c. $\dfrac{1}{2}-\ln 2$ d. $1-\ln 2$ e. None of the above

6. If f is differentiable at 2, find $\lim_{x\to 2} \dfrac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}}$

   a. *f(2)* b. $2\ f'(2)$ c. $\dfrac{\sqrt{2}\ f'(2)}{2}$ d. $2\sqrt{2}\ f'(2)$ e. Does not exist

7. Let $\dfrac{2+\sin x}{y+3}\left(\dfrac{dy}{dx}\right)=-\cos x,\quad y(0)=1.$ Find $y'(0)$.

   a. -2 b. -4/3 c. -3/2 d. 0 e. None of these

8. Find the derivative of tan x with respect to sin x.

a. sec x tan x b. $\sec^2 x$ c. $\sec^2 x \sin x$ d. $\sec^2 x \cos x$ e. $\sec^3 x$

9. Let $y = \dfrac{x}{1+|x|}$. Find $y'(0)$.

a. -1 b. 0 c. ½ d. 1 e. Does not exist

10. $xy = (x+y)^n$ and $\dfrac{dy}{dx} = \dfrac{y}{x}$. Find the value of n.

a. 0 b.3 c.4 d. 5 e. None of the above

11. Let a < 0 < b. Evaluate $\int_a^b \dfrac{|x|}{x}\,dx$

a. a + b b. a - b c. b - a d. Does not exist e. None of the above

12. Evaluate $\displaystyle\int \frac{x^5}{1+x^2}\,dx$

a. $\dfrac{x^4}{4} - \dfrac{x^2}{2} + \ln\sqrt{1+x^2} + c$

b. $\dfrac{x^4}{4} - \dfrac{3x^2}{2} + \ln\sqrt{1+x^2} + c$

c. $\dfrac{x^4}{4} + \dfrac{3x^2}{2} + \ln\sqrt{1+x^2} + c$

d. $\dfrac{x^4}{4} + \dfrac{x^2}{2} + \ln\sqrt{1+x^2} + c$

e. None of the above.

13. Evaluate $\int \frac{\sin^4 x + \cos^4 x}{\sin^3 x \cos^3 x}\, dx$

a. $\frac{\sec^2 x}{2} - \frac{\csc^2 x}{2} + c$

b. $-\frac{\sec^2 x}{2} + \frac{\csc^2 x}{2} + c$

c. $-4\sec^3 x + 4\csc^3 x + c$

d. $4\sec^3 x - 4\csc^3 x + c$

e. None of these

14. Evaluate: $\int x^5 \sqrt{1+x^3}\, dx$

a. $\frac{5(1+x^3)^{5/2}}{6} - \frac{(1+x^3)^{3/2}}{2} + c$

b. $-\frac{5(1+x^3)^{5/2}}{6} + \frac{(1+x^3)^{3/2}}{2} + c$

c. $-\frac{2(1+x^3)^{5/2}}{15} + \frac{2(1+x^3)^{3/2}}{9} + c$

d. $\frac{2(1+x^3)^{5/2}}{15} - \frac{2(1+x^3)^{3/2}}{9} + c$

e. None of the above

15. Evaluate $\int_0^1 \frac{\sin(x)}{\sin(x) + \sin(1-x)}\, dx$

a. $\frac{\cos 1 - 1}{\cos 1 + 1}$

b. $\frac{\cos 1 + 1}{\cos 1 - 1}$

c. $\frac{\cos 1}{\cos 1 + 1}$

d. $\frac{\cos 1}{\cos 1 - 1}$

e. None of these

16. Suppose f is continuous, f(0) = 0, f(1) = 1, $f'(x) > 0$ and $\int_0^1 f(x)dx = 1/3$.

What is the value of $\int_0^1 f^{-1}(y)dy$?

a. 3   b. 1   c. 1/3   d. Not enough information   e. None of the above

17. Find the equation of the tangent line to $y = x^2 + 1$ which is parallel to the line y = -4x.
   a. 4x + y + 3 = 0
   b. x + 4y + 3 = 0
   c. 4x +y + 4 = 0
   d. x + 4y + 4 = 0
   e. None of the above

18. Tangent lines at the points $(at^2, 2at)$ and $(as^2, 2as)$ on the parabola $y^2 = 4ax$ are perpendicular. Then which one of the following is true:

   a. a > 0
   b. a >0 and st = -1
   c. st = -1
   d. a >0 and st = 1
   e. None of the above

19. The area enclosed between the curves $y^2 = ax$ and $x^2 = ay$ (a >0) is 1 square unit. What is the value of a?

a. $\sqrt{2}$ b. $\sqrt{5}$ c. $\sqrt{6}$ d. $\sqrt{7}$ e. None of these

20. The function $f(x) = xe^{-x}$ has a local maximum at x equals:

a. 0   b. e^{-1} c. 1   d. e   e. None of the above

21. If f(1) = 10, and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can f(4) possibly be?

a. 8   b. 10   c. 14   d. 16   e. None of these

22. Suppose f is continuous on [0,1] and f(0) = 6 and f(1) = -2. From the Intermediate Value Theorem for continuous functions, it follows that:

   a. f(1/2) = 2
   b. There is a unique c in (0,1) where f(c) = 5
   c. There is a value c in between 0 and 1 where f(c) equals the circumference of circle of radius 1/2
   d. There is a value c between 0 and 1 where f(c) equals 6 or -2
   e. None of the above

23. The solutions of $2xy\,\dfrac{dy}{dx} = (1 + y^2)$ are given:

   a. $3y^2 = cx\,(3+y^3)$, c is a constant
   b. $1+y^2 = cx$, c is a constant
   c. $1+ y^2 = cx^2$, c is a constant
   d. $3y^2 = cx\,(3 + x^3)$, c is a constant
   e. None of the above

24. A particle moves along the curve $y = x^2 + 2x$. Find the Cartesian coordinates of the point on the curve where the rate of change for the x and y coordinates of the particle is the same.

   a. (1/2, 3/4) b(-1/2, 3/4) c. (1/2, -3/4) d. (-1/2, -3/4)

   e. None of these

Name ____

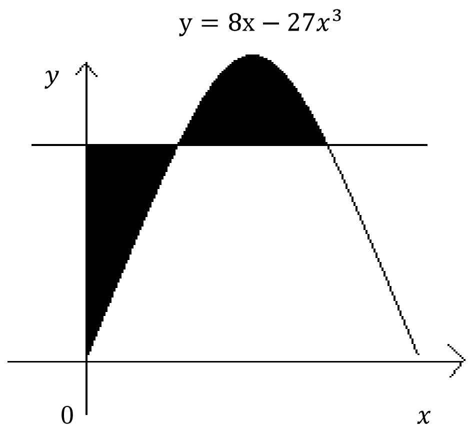
Tie Breaker #1

There is a line through the origin that divides the region bounded by the parabola y= $x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

Name ____

Tie Breaker #2

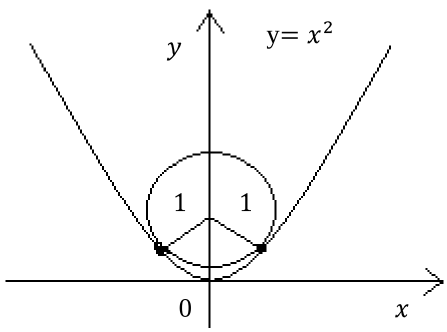
The figure shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.



Name ____

Tie Breaker #3

The figure shows a circle with radius 1 inscribed in the parabola y= x^2. Find the center of the circle.

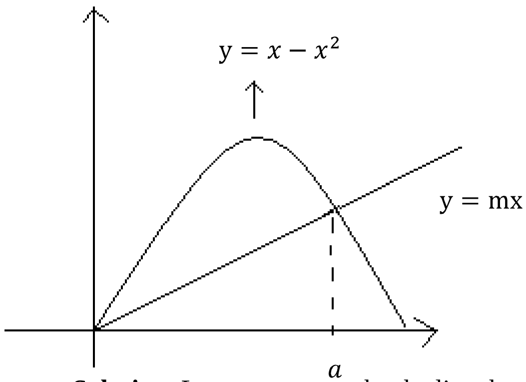


Arkansas Council of Teachers of Mathematics
Calculus Exam Spring 2009
Answer Key [NOTE: Revised with only 24 questions]

1. b
2. c
3. c
4. b
5. e
6. d
7. a
8. e
9. d
10. e
11. a
12. a
13. a
14. d
15. e
16. e
17. a
18. c
19. e
20. c
21. d
22. c
23. b
24. d

Tie Breaker #1

There is a line through the origin that divides the region bounded by the parabola y$= x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?



Solution: Let $y = mx$ be the line that divides the area bounded by the parabola and x-axis into two regions with equal area. Let a be as denoted in the above figure. First, note that the area bounded by the parabola and the x-axis is $\int_0^{1} (x - x^2)dx = \frac{1}{6}$.

Second, $ma = a - a^2$ which implies $m = 1 - a$.

Since the line divides the area into two halves, we have

$$\int_0^a (x - x^2 - mx)dx = \frac{1}{12}$$

(or) $$\left[\frac{x^2}{2} - \frac{x^3}{3} - m\frac{x^2}{2}\right]_0^a = \frac{1}{12}$$

i.e. $$\frac{a^2}{2} - \frac{a^3}{3} - m\frac{a^2}{2} = \frac{1}{12}$$

$$\Rightarrow \quad \frac{a^2}{2} - \frac{a^3}{3} - (1 - a)\frac{a^2}{2} = \frac{1}{12} \quad (since\ m = 1 - a)$$

$$\Rightarrow \quad -\frac{a^3}{3} + \frac{a^3}{2} = \frac{1}{12}$$

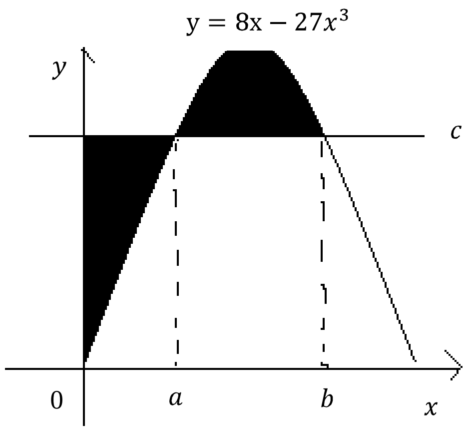
$$\Rightarrow a^3 = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{\sqrt[3]{2}}$$

Therefore, $m = 1 - \frac{1}{\sqrt[3]{2}}$.

Tie Breaker #2

The figure shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.



Solution: Let a, b be as noted in the above figure. Obviously $8a - 27a^3 = c$ and $8b - 27b^3 = c$. Since the shaded regions have equal area,

$$\int_0^a [c - (8x - 27x^3)]dx \quad = \quad \int_a^b (8x - 27x^3 - c)dx$$

$$cx - 4x^2 + \frac{27x^4}{4}\Big|_0^a \quad = \quad 4x^2 - \frac{27x^4}{4} - cx\Big|_a^b$$

i.e., $ca - 4a^2 + \frac{27a^4}{4} \quad = \quad 4b^2 - \frac{27b^4}{4} - cb - 4a^2 + \frac{27a^3}{4} + ca$

$$\Rightarrow \qquad 4b^2 - \frac{27b^4}{4} - cb = 0$$

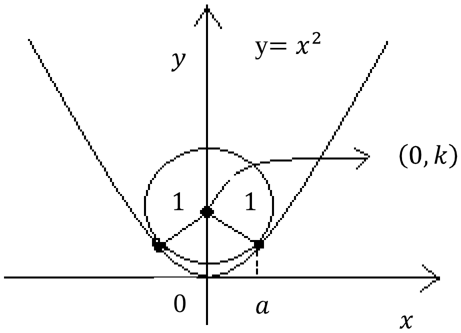
$$\Rightarrow \qquad 4b - \frac{27b^3}{4} = c \quad = \quad 8b - 27b^3$$

$$\Rightarrow\ 81b^3 - 16b = 0 \quad \Rightarrow \quad b = \frac{4}{9}$$

Therefore $c = 8b - 27b^3 = 8 \times \frac{4}{9} - 27\left(\frac{4}{9}\right)^3 = \frac{32}{27}$

Tie Breaker #3

The figure shows a circle with radius 1 inscribed in the parabola y= x^2. Find the center of the circle.



Solution: Let $(0, k)$ be the center of the circle, and a as shown in the figure. Since the circle is inscribed in the parabola, the tangent line to the circle as well as the parabola are the same. Therefore, $\frac{dy}{dx}$ at the point $(a,\ a^2)$ is the same for the circle and the parabola. This means

$$-\frac{x}{y-k}\ @(a, a^2) = 2x\ @(a, a^2)$$

$$\text{i.e., } -\frac{a}{a^2-k} = 2a$$

$$\Rightarrow\ a^2 - k = -\frac{1}{2}\ \_\_\_\_\_\_\_\_\_\_\_\ \ 1$$

Also, since the radius of the circle is 1, we have

$$a^2 + (a^2-k)^2 = 1$$

$$\Rightarrow\ a^2 + \frac{1}{4} = 1 \qquad \text{(by } 1\text{)}$$

$$\Rightarrow\ a^2 = \frac{3}{4}$$

Therefore by 1 , $k = \frac{5}{4}$.