

Arkansas Council of Teachers of Mathematics
2012 State Competition
Calculus Exam

For questions 1 through 25, mark your answer choice on the answer sheet provided. Unless otherwise stated, assume all variables are real and all functions are continuous over relevant domains. Assume all angles are in radians. After completing items 1 through 25, answer each of the tiebreaker items in sequential order (do #1 first, followed by #2, and then #3 last). Be sure that your name is printed on each of the tiebreaker pages. Congratulations for being selected to participate in the ACTM State Contest!

1. Consider the following limit:

$$\lim_{x \rightarrow 3} 4x - 2$$

Given $\epsilon = .05$, determine the value for δ to satisfy the $\epsilon - \delta$ definition of the limit.

- a. 0
 - b. .0125
 - c. .05
 - d. 10
 - e. Cannot be determined.
2. Determine the slope of the normal line to the function $\phi(\theta) = \ln(\cos(\theta) + 1)$ at the point $\theta = \pi/2$.
- a. $m_{norm} = -8$
 - b. $m_{norm} = -1/8$
 - c. $m_{norm} = 0$
 - d. $m_{norm} = 1$
 - e. $m_{norm} = 8$
3. The US Department of Energy Statistics found that the rate of consumption of gasoline in the US is approximated by

$$\frac{dA}{dt} = 2.74 - .11t + .01t^2$$

with $t = 0$ corresponding to 1980, and A is in millions of gallons. Determine the number of gallons consumed between 1980 and 1985. Round your answer to the nearest one hundredth.

- a. 2.44 million gallons
- b. 2.74 million gallons
- c. 12.74 million gallons
- d. 39,186.64 million gallons.
- e. Cannot be determined

4. The equation of the tangent line(s) to the curve $x = y^2$ at $x = 4$ is

- a. $y = \frac{x}{4} + 1$
- b. $y = -\frac{x}{4} - 1$
- c. $4x + 16y = -16$
- d. a, b and c
- e. none of these

5. Determine $f'(x)$ by differentiating. Don't simplify.

$$f(x) = (7x + \sqrt{x^2 + 3})^6$$

- a. $f(x) = 0$
- b. $f(x) = 6(7x + \sqrt{x^2 + 3})^5 \cdot (7 + \frac{1}{2}(x^2 + 3)^{-1/2}) \cdot (2x)$
- c. $f(x) = 6(7 + \frac{1}{2}(x^2 + 3)^{-1/2} \cdot (2x))^5$
- d. $f(x) = 6(7x + \sqrt{x^2 + 3})^5 \cdot (7 + \frac{1}{2}(x^2 + 3)^{-1/2} \cdot (x^2 + 3) \cdot (2x))$
- e. $f(x) = 6(7x + \sqrt{x^2 + 3})^5 \cdot (7 + \frac{1}{2}(x^2 + 3)^{-1/2} \cdot (2x))$

6. Determine all values of x where the function $f(x)$ and its derivative intersect.

$$f(x) = 2x^2 - 3x$$

- a. $x = -\frac{3}{4}$
- b. $x = 3$ or $\frac{1}{2}$
- c. $x = -\frac{3}{2}$ or 0
- d. $x = -1$ or $\frac{3}{2}$
- e. No intersection points

7. Find the global maximum to the following function on the given domain.

$$f(x) = 5 - 6x^2 - 2x^3 \quad \text{on the interval } [-4, 1]$$

- a. $y = -3$
- b. $y = 5$
- c. $y = 37$
- d. $y = 45$
- e. There is no maximum

8. Given $f(x) = x^4 - 4x^3 + 4x^2$ on the interval $[-1, 3]$, then the value(s) of c such that

$$f'(c) = \frac{f(3) - f(-1)}{3 - (-1)} \text{ is/are}$$

- a. $c = 0$
- b. $c = 1$
- c. $c = 2$
- d. all of these
- e. none of these

9. Determine the minimum number of critical numbers that a cubic function of the form $f(x) = ax^3 + bx^2 + cx + d$ may have.

- a. 0
- b. $-\frac{b}{2a}$
- c. $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- d. 1
- e. 2

10. Find all values of x where the following function is continuous.

$$f(x) = \frac{x}{\sqrt{4 - x^2}}$$

- a. $x \neq 0$
- b. $x = 0$
- c. $-2 \leq x \leq 2$
- d. $-2 < x < 2$
- e. $x \leq -2$ or $2 \leq x$

11. Evaluate the following integral

$$\int e^x \cos(x) \, dx$$

- a. $e^x \cdot \sin(x) + C$
- b. $-e^x \cdot \sin(x) + C$
- c. $\frac{1}{2}e^x \cdot \cos(x) + C$
- d. $\frac{1}{2}e^x \cdot (\sin(x) + \cos(x)) + C$
- e. $-\frac{1}{2}e^x \cdot (\sin(x) + \cos(x)) + C$

12. The values of a and b , respectively, which make the function $f(x) = \begin{cases} x^3 + ax^2 + 1, & x \leq 1 \\ -x^2 + bx + 2, & x > 1 \end{cases}$

both continuous and differentiable at $x = 1$ are

- a. -4 and -3
- b. -1 and 0
- c. -1 and 1
- d. -3 and -1
- e. None of these

13. What is/are the region(s) where the following function is concave down on the domain $[0, 2\pi]$?

$$f(x) = \sin x$$

- a. $(\frac{\pi}{2}, \frac{3\pi}{2})$
- b. $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$
- c. $(0, \pi)$
- d. $(\pi, 0)$
- e. The function has no regions where it is concave down.

14. Solve the differential equation with the given initial conditions.

$$\frac{d^2y}{dx^2} = x^3 - 5, \quad y'(2) = -1 \text{ and } y(0) = 2$$

- $y = 6x$
- $y = 3x^2$
- $y = \frac{x^5}{20} - \frac{5x^2}{2}$
- $y = \frac{x^5}{20} - \frac{5x^2}{2} + 5x + 2$
- Cannot be determined

15. The relationship between the temperature F on the Fahrenheit scale and the temperature C on the Celsius scale is given by $C = \frac{5}{9}(F - 32)$. Find the Rate of Change of F with respect to C .

- $\frac{dC}{dF} = \frac{5}{9}$
- $\frac{dF}{dC} = \frac{9}{5}$
- $\frac{dC}{dF} = 0$
- $\frac{dF}{dC} = 0$
- $F = \frac{9}{5}C + 32$

16. Calculate the volume of the solid of revolution generated by the region bounded by the curve $x = \sqrt{25 - y^2}$ and the lines $x = 0$ and $x = 3$, rotated around the y -axis. Use the disk method.

- $V = 4$
- $V = \frac{244}{3}\pi$
- $V = \frac{472}{3}\pi$
- $V = \frac{500}{3}\pi$
- none of these

17. Determine the sum of the following:

$$\sum_{i=1}^{21} (5i^2 - 2)(i - 4)$$

- 99,078
- 99,708
- 200,291
- 200,299
- The summation has no value

18. Rolle's Theorem applies to the function $f(x) = 4x^2 - 20x + 29$ on the interval $[1,4]$. Find the value(s) of c that satisfy the conclusion of Rolle's Theorem.

- $c = 0$
- $c = \frac{3}{2}$
- $c = \frac{5}{2}$
- $c = 13$
- $c = 1, 4$

19. Estimate the area under the curve bounded by $5x + 4y = 20$, the x -axis, and the y -axis, using four rectangles of equal width. Use right hand end points.

- a. $A = 4.375$
- b. $A = 7.5$
- c. $A = 9.375$
- d. $A = 10$
- e. $A = 12.5$

20. Evaluate the following limit:

$$\lim_{x \rightarrow -3} \frac{x^3 - 4x^2 - 21x}{x + 3}$$

- a. -9
- b. 0
- c. 1
- d. 30
- e. the limit does not exist

21. Determine the range of the following function:

$$G(x) = \frac{x + 2}{x^2 - 4}$$

- a. $\{y | y \in 4\}$
- b. $\{y | y \neq 0\}$
- c. $\{y | y \neq -2, y \neq 2\}$
- d. $\{y | y \neq -2, y \neq 0, y \neq 2\}$
- e. $\{y | y \neq -.25, y \neq 0\}$

22. Use implicit differentiation to find the slope of the tangent line to the function at the indicated point.

$$x^2y + \sin(y) = 2\pi \quad \text{at the point } (1, 2\pi)$$

- a. $m_{tan} = 0$
- b. $m_{tan} = \frac{1}{2\pi}$
- c. $m_{tan} = -\frac{1}{2\pi}$
- d. $m_{tan} = 2\pi$
- e. $m_{tan} = -2\pi$

23. Use the Fundamental Theorem of Calculus to evaluate the following

$$\frac{d}{dx} \int_0^x \sqrt{t^4 + t} dt$$

- a. $\sqrt{x^4 + x}$
- b. 0
- c. $\sqrt{t^4 + t}$
- d. x
- e. Cannot be evaluated

24. Consider the function,

$$g(x) = \begin{cases} \frac{1}{x+1} & \text{if } x < -1 \\ \frac{\sin(x)}{x} & \text{if } -1 \leq x < 0 \\ \ln(x) & \text{if } x > 0 \end{cases}$$

How many vertical asymptotes does this function have?

- a. 1
- b. 2
- c. 3
- d. 4
- e. It has no vertical asymptotes.

25. Apply the Mean Value Theorem for Definite Integrals to the following function.

$$f(x) = x^2 + 2x - 1 \quad \text{on the interval } [0,3]$$

Determine the value z , such that

$$\int_a^b f(x) dx = f(z)(b - a)$$

- a. $z = 5$
- b. $z = 3$
- c. $z = -1 + \sqrt{7}$
- d. $z = -1 - \sqrt{7}$
- e. None of the above

Name _____

School _____

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Tiebreaker Questions

Your solutions should be written clearly. All work leading to your final answer must be included. The questions will be used in sequential order to resolve ties for first, second, and/or third place.

Tiebreaker #1.

Solve the following integration problem.

$$\int \frac{x^3 + 8x^2 + 16x - 4}{x + 5} dx$$

Name _____

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Tiebreaker #2.

If $f(x) = x^{(x)^x}$. Find $f'(x)$.

Name _____

School _____

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Tiebreaker #3.

The Gateway Arch in St. Louis has the shape of an inverted catenary. The shape is approximated by the function

$$A(x) = -127 \cosh\left(\frac{x}{127.7}\right) + 757.7 \quad -315.6 \leq x \leq 315.6$$

where the hyperbolic cosine function, $\cosh(x) = \frac{e^x + e^{-x}}{2}$

Approximate the total open area under the arch.

**Arkansas Council of Teachers of Mathematics
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Solutions**

Multiple Choice Answers

1.....B

2.....D

3.....C

4.....D

5.....E

6.....B

7.....C

8.....D

9.....A

10....D

11....D

12....A

13....C

14. ..D

15. ..B

16. ..B

17. ..C

18. ..C

19. ..B

20. ..D

21. ..E

22. ..E

23. ..A

24. ..B

25. ..C

Tie Breaker Question 1 Solution

1. Solve the following integration problem.

$$\int \frac{x^3 + 8x^2 + 16x - 4}{x + 5} dx$$

At first glance this doesn't look solvable. Using long division reduces the problem into something manageable. By long division,

$$\frac{x^3 + 8x^2 + 16x - 4}{x + 5} = x^2 + 3x + 1 - \frac{9}{x + 5}$$

Thus,

$$\begin{aligned} \int \frac{x^3 + 8x^2 + 16x - 4}{x + 5} dx &= \int x^2 + 3x + 1 - \frac{9}{x + 5} dx \\ &= \frac{x^3}{3} + \frac{3x^2}{2} + x - 9 \ln(|x + 5|) + C \end{aligned}$$

Tie Breaker Question 2 Solution

2. If $f(x) = x^{(x)^x}$. Find $f'(x)$.

$$f'(x) = x^{x^x+x-1} (x * (\ln(x))^2 + x * \ln(x) + 1), x > 0$$

Tie Breaker Question 3 Solution

3. The Gateway Arch in St. Louis has the shape of an inverted catenary. The shape is approximated by the function

$$A(x) = -127 \cosh\left(\frac{x}{127.7}\right) + 757.7 \quad -315.6 \leq x \leq 315.6$$

where cosh is the hyperbolic cosine function.

- a. Approximate the total open area under the arch.

The key to this problem is $\int \cosh(x) dx = \sinh(x) + C$.

$$\begin{aligned} \int_{-315.6}^{315.6} -127 \cosh\left(\frac{x}{127.7}\right) + 757.7 dx &= -127 \cdot 127.7 \cdot \sinh\left(\frac{x}{127.7}\right) + 757.7x \Big|_{-315.6}^{315.6} \\ &= 143811.43 - (-143811.43) = 287622.87 \text{ units}^2 \end{aligned}$$