Arkansas Council of Teachers of Mathematics

2014 Regional Exam

Pre-Calculus

For questions 1 through 25, mark your answer choice on the answer sheet provided. After completing items 1 through 25, answer each of the tiebreaker items in sequential order (do #1 first, followed by #2, and then #3 last). Be sure that your name is printed on each of the tiebreakers.

1. How many solutions does $4^x = x^4$ have?

   A. 0
   B. 1
   C. 2
   D. 3
   E. Each of the other answers is incorrect.

2. What is the range of $f(x) = \text{arccot}(x)$?

   A. $(-\infty, \infty)$
   B. $(-\pi/2 , \pi/2)$
   C. $(0, \pi)$
   D. $[-\pi/2, \pi/2]$
   E. $[0, \pi]$

3. Solve the following system of equations:

$$\begin{aligned} 3x + 2y - z &= 4 \\ -1x + 3y - 2z &= -1 \\ 2x + y + z &= 7 \end{aligned}$$

   The x coordinate of the solution is:
   A. 0
   B. 1
   C. 2
   D. 3
   E. Each of the other answers is incorrect.

4. Given the parametric equations: $x = 3\cos(t)$, $y = 4\sin(t)$ for $t \in [0, 2\pi]$ which of the following best describes the graph?

   A. Line
   B. Line segment
   C. Circle
   D. Ellipse
   E. Arc of a circle

5. The following gives temperatures and times at a certain city in Arkansas on a certain day.

Time	1:00 pm	2:00 pm	3:00 pm
Temperature	47°F	40°F	32°F

For the interval from 1:00 to 3:00 what is the average rate of change in the temperature?

A. -15°F
B. -15°F/hour
C. -5°F/hour
D. -7.5°F/hour
E. Each of the other answers is incorrect.

6. What are the vertical asymptotes of $f(x) = \frac{x+2}{x^2-4}$?

A. $x = 2$
B. $x = -2$
C. $x = 2$ and $x = -2$
D. $x = 0$
E. Each of the other answers is incorrect.

7. Find all the solutions on the interval $[0,2\pi]$ to $2\sin(x) + 1 = \csc(x)$. Choose the answer below that is the **sum** of these solutions.

A. $\frac{\pi}{6}$
B. $\frac{5\pi}{2}$
C. $\frac{5\pi}{3}$
D. π
E. $\frac{3\pi}{2}$

8. Consider the parallelogram given by the two vectors $\vec{u} =< \sqrt{5}, 2 >$ and $\vec{v} =< 3\sqrt{5}, 2 >$. Find the ratio of perimeter to the area of the parallelogram.

A. $4\sqrt{5}$
B. $2\sqrt{5}$
C. $\frac{\sqrt{5}}{2}$
D. 20
E. $\sqrt{5}$

9. A rectangle whose width is three times its length is inscribed inside a circle given by $x^2 - 2x + y^2 - 6y = 15$. Find the area of the rectangle.

   A. $\frac{75}{4}$ units squared
   B. 30 units squared
   C. $6\sqrt{10}$ units squared
   D. $\frac{15}{2}$ units squared
   E. 75 units squared

Questions 10 and 11: A fishing boat leaves port at 7:00 am and travels due east at 20 mph for 2 hours. It then heads due south traveling 10 mph. A supply boat leaves the same port at 9:00 am traveling 20 mph to take some forgotten supplies to the fishing boat. Assume they take the most direct route possible to the fishing boat in the problems below.

10. How long will it take for the supply boat to reach the fishing boat? Round this answer to the nearest minute.

    A. 2 hours and 44 minutes
    B. 2 hours and 19 minutes
    C. 2 hours and 17 minutes
    D. 2 hours and 31 minutes
    E. 2 hours and 10 minutes

11. How far did the supply boat travel?

    A. 55 miles
    B. 50 miles
    C. 46 miles
    D. 43 miles
    E. 57 miles

12. Three funds grow exponentially. One grows at an annual rate of 5 percent, the second at an annual rate of 7.5 percent, and the third at an annual rate of 9.8 percent. The same amount of money is initially invested in all three funds. How long will it take for the overall sum of money to double? Round this answer to two decimal places.

    A. 3.22 years
    B. 14.21 years
    C. 9.58 years
    D. 9.48 years
    E. Each of the other answers is incorrect.

13. Consider $f(x) = e^x$ and $g(x)$ defined by the table below.

x	$g(x)$
1	ke^{-c}
$\ln(c)$	1

Let $h(x) = (f^{-1} \circ g)(x) + (f \circ g^{-1})(x)$. Find $h(1)$.

A. ke^c
B. $e^{-1} + c$
C. $ke^{-c} + \ln(c)$
D. $\ln(k)$
E. $e^{ke^{-(c)}} + \ln(c)$

Questions 14 and 15: For a function *f* the *symmetric difference quotient* is defined to be

$$\frac{f(x+h) - f(x-h)}{2h}.$$

14. Which of these is the best interpretation of the symmetric difference quotient?

A. If *h* is close to 0 the symmetric difference quotient is close to 0.
B. An approximation of the output value of the function at $x = h$.
C. The symmetric difference quotient is the slope between the following two points (x, $f(x)$) and ($x + h$, $f(x + h)$).
D. The symmetric difference quotient is the slope between the following two points ($x + h$, $f(x + h)$) and ($x - h$, $f(x - h)$).
E. The symmetric difference quotient is the slope of the line tangent to the graph of the function at (x, $f(x)$).

15. Find the symmetric difference for $f(x) = 3x^2 - 5$.

A. 0
B. 6
C. $6x$
D. $6x - 5$
E. $6x - 5h$

16. A historian has a cotton uniform that he claims was made for General George Washington. A sample of the uniform is taken and dated using C-14 dating. It was determined that 97.15% of the original carbon 14 is still remaining. Carbon 14 decays exponentially with a half-life of 5730 years. What can we say about the historian's claim?

A. This is definitely not General Washington's uniform because it is made of cotton that was alive much after he was a general.
B. This is definitely not General Washington's uniform because it is made of cotton that died way earlier than he was a general.
C. This is definitely General Washington's uniform because it is made of cotton that was harvested during the Revolutionary War.
D. This could be General Washington's uniform because it is made of cotton that was harvested during the Revolutionary War.
E. None of the claims above can be made.

Questions 17 and 18: A population of bacteria is modeled closely by the following function where *t* is time in days

$$P(t) = \frac{2364}{1+2e^{-0.02t}}.$$

17. How many bacteria were in the initial population?

A. 1
B. 2
C. 788
D. 1182
E. 2364

18. After a long period of time how many bacteria will eventually be in the population?

A. 0
B. 1
C. 2364
D. 4728
E. The population will approach infinity.

19. A carpenter measures 3.00 feet out horizontally along a wall and marks point A. Then from the same point she measures 4.00 feet out horizontally along the adjacent wall marking point B. She then measures the distance between points A and B to be 4.83 feet. To the nearest degree what angle is formed between the two walls?

A. 81^o
B. 86^o
C. 90^o
D. 95^o
E. Each of the other answers is incorrect.

20. Two precalculus students stand on one side of river at points *A* and *B*. Points *A* and *B* are 165 feet apart. Each one sights the same tree on the opposite side of the river. From their respective locations they determine the angles between the tree and the other person to be 75° and 36° respectively. To the nearest foot how wide is the river at this point?

A. 100 feet
B. 90 feet
C. 75 feet
D. 120 feet
E. Each of the other answers is incorrect.

21. How many solutions are there to the following equation?

$$\sin^3(2x)+\sin^2(2x)-2\sin(2x)=0$$

A. 0
B. 1
C. 2
D. 3
E. Each of the other answers is incorrect.

22. $i^{235,268,478,225,125,136,253} =$

A. 1
B. -1
C. i
D. $-i$
E. Each of the other answers is incorrect.

23. $\dfrac{1}{\sqrt[3]{x}-1} =$

A. $\dfrac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{x-1}$

B. $\dfrac{\sqrt[3]{x}+1}{x-1}$

C. $\dfrac{\sqrt[3]{x}-1}{x-1}$

D. $\dfrac{\sqrt[3]{x^2}}{x-1}$

E. Each of the other answers is incorrect.

24. $\sqrt{x^2-8x+16} =$

A. $x-4$

B. $x+4$

C. $|x+4|$

D. $|x-4|$

E. Each of the other answers is incorrect.

25. $\cos\left(\arctan\left(\frac{x}{5}\right)\right) =$

A. $\frac{5}{\sqrt{x^2+25}}$

B. $\frac{5}{\sqrt{x^2-25}}$

C. $\frac{x}{\sqrt{x^2+25}}$

D. $\frac{x}{5}$

E. Each of the other answers is incorrect.

Name:_____

Tie-Break 1

The sales volume data for a company is recorded in the table below.

Year	1980	1990	2000	2005	2010
Items Sold	1025	12,490	152,200	530,920	1,853,250

A. Would the data be best approximated by a linear, quadratic, or exponential function?

B. Use the regression capabilities of your calculator to obtain a formula of the type you selected which best models the items sold as a function of the year.
t = year, S = items sold

C. Use your model formula to predict the volume of sales in 2016.

Name:_____

Tie-Break 2

A scientist measured the depth of the water at a certain location and noticed that there was a periodic variation in the depth which could be modeled closely with a formula of the form $y = y_0 + A\cos[Bt]$, where t is in hours since midnight on January 1, 2000. He recorded the level at high tide to be 24 meters and the depth at low tide to be 9 meters. The time between successive high tides is 12.4 hours. Low tide occurred at midnight on January 1, 2000.

A. What is the physical meaning of y_0? What is its value?

B. What is the physical meaning of A? What is its value?

C. What is the value of B?

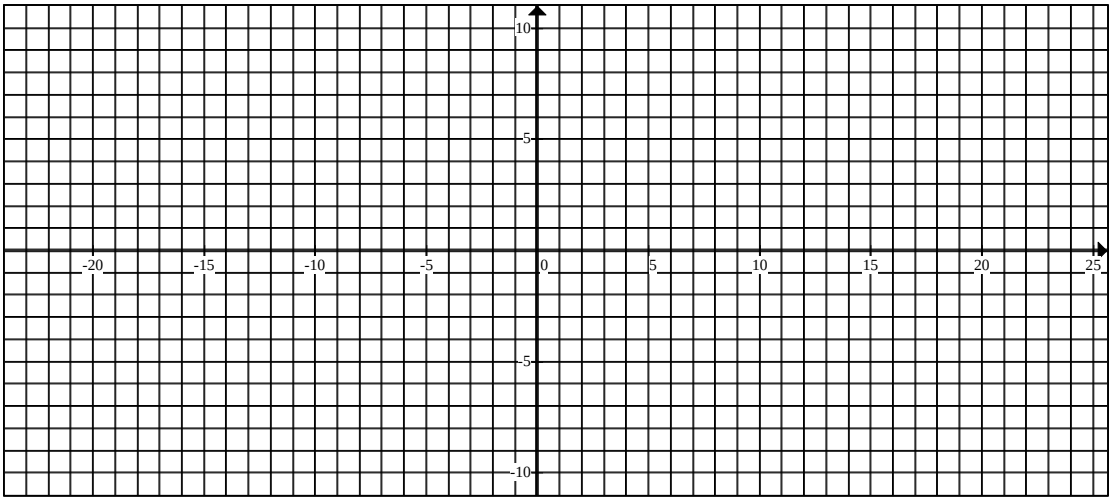
Name:_____

Tie-Break 3

Give an equation for an ellipse centered at (0, 2) with vertical radius 8 and horizontal radius 4.

A. Give an equation for a hyperbola with asymptotes $y = x$ and $y = -x$ and vertices (0,1) and (0, -1) .

B. Graph these two conic sections on the grid below.



C. Find the exact coordinates of all intersections of the two graphs.

ANSWERS:

1. D
2. C
3. B
4. D
5. D
6. A
7. B
8. E
9. B
10. B
11. C
12. D
13. D
14. D
15. C
16. D
17. C
18. C
19. B
20. A
21. E (There are infinitely many solutions.)
22. C
23. A
24. D
25. A

ACTM Regional Precalculus Competition 2014
Tie Breaker 1

Name **Solution**

The sales volume data for a company is recorded in the table below.

Year	1980	1990	2000	2005	2010
Items Sold	1025	12,490	152,200	530,920	1,853,250

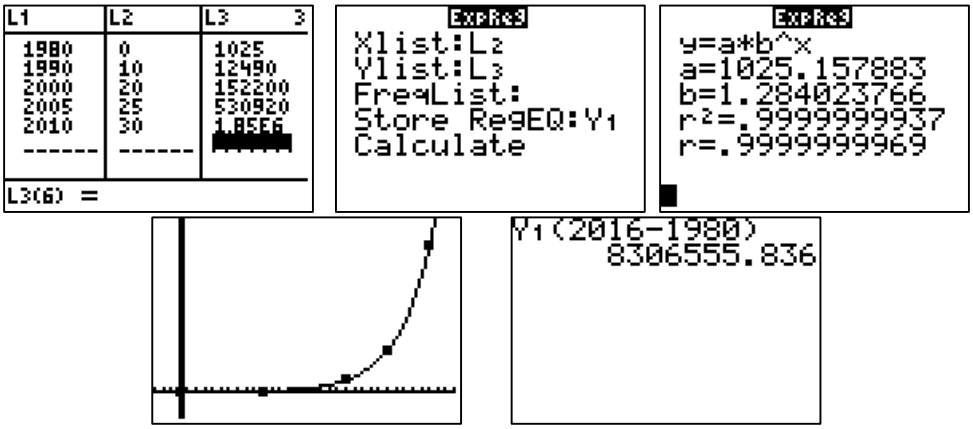
A. Would the data be best approximated by a linear, quadratic, or exponential function?
An **exponential function** will model this data extremely closely, much more closely than a linear or quadratic function.

B. Use the regression capabilities of your calculator to obtain a formula of the type you selected which best models the items sold as a function of the year.
t = year, S = items sold

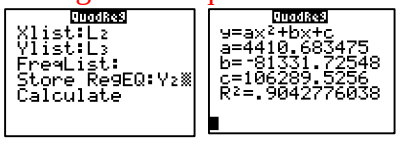
$$S = 1025.157883(1.284023766)^{(t-1980)}$$

C. Use your model formula to predict the volume of sales in 2016.
The model predicts 8,306,556 items sold in 2016.

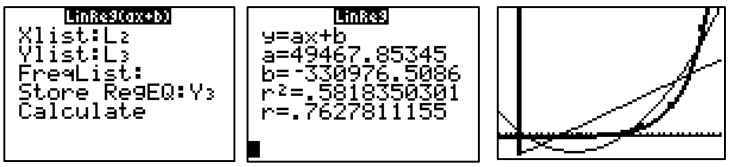
Calculator work:
L1 = t = year, L2 = x = year – 2008, L3 = y = S= items sold



Not as good is a quadratic model:



or a linear model:



ACTM Regional Precalculus Competition 2014
Tie Breaker 2

Name **Solution**

A scientist measured the depth of the water at a certain location and noticed that there was a periodic variation in the depth which could be modeled closely with a formula of the form $y = y_0 + A\cos[Bt]$, where t is in hours since midnight on January 1, 2000. He recorded the level at high tide to be 24 meters and the depth at low tide to be 9 meters. The time between successive high tides is 12.4 hours. Low tide occurred at midnight on January 1, 2000.

A. What is the physical meaning of y_0? What is its value?
In general, y_0 is the centerline or average value. In this problem this is the **average depth of the water** at this spot. It will be halfway between the high and low tide. Low tide is 9 meters. High tide is 24 meters. Just average the high and low values to find the centerline: (9+24)/2 = 33/2 = **16.5m**.

B. What is the physical meaning of A? What is its value?
In general $|A|$ is the amount of vertical stretch and is called the amplitude. In this context it is **one-half of the difference between the depths at high and low tide**. We computed it to be (24-9)/2 = 15/2 = 7.5 meters = $|A|$. Since a standard cosine wave begins at a maximum and there no phase shift we take A to be negative so that the graph will start at a minimum. Therefore, ***A* = -7.5**.

C. What is the value of B?
In general, $B = \frac{2\pi}{P}$ where P is the period. In this problem the period (length of time for one cycle) is the time between successive high tides, which is given to be 12.4 hours = P. Therefore, $B = \frac{2\pi}{12.4}$.

ACTM Regional Precalculus Competition 2014
Tie Breaker 3

Name **Solution**

A. Give an equation for an ellipse centered at (0, 2) with vertical radius 8 and horizontal radius 4.

$$\frac{x^2}{16}+\frac{(y-2)^2}{64}=1$$

$$4x^2+(y-2)^2=64$$

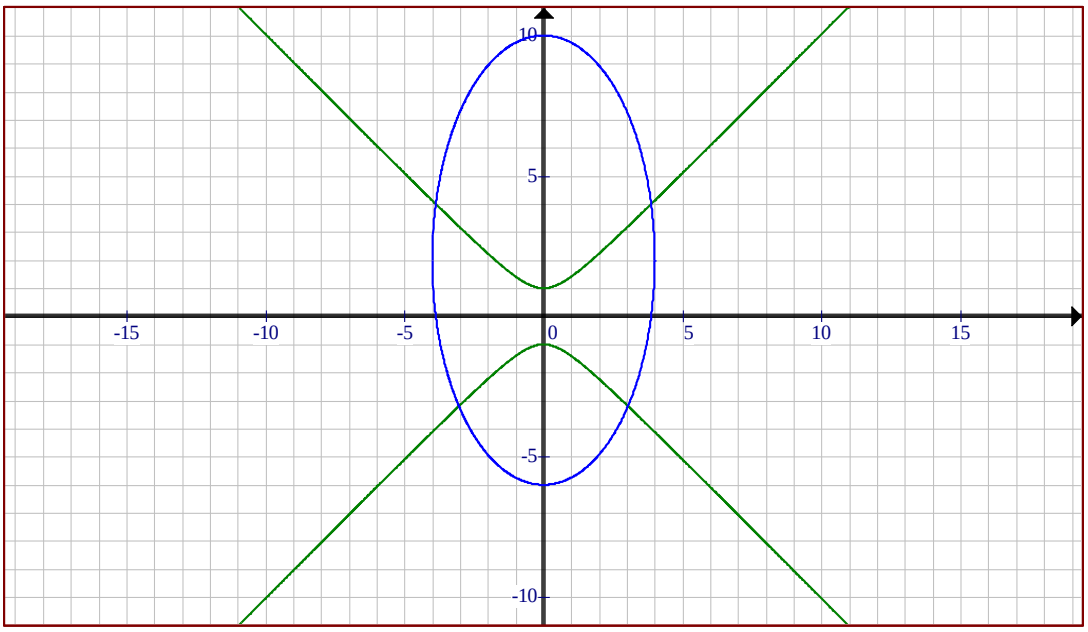
$$4x^2+(y-2)^2=64 \Rightarrow (y-2)^2=64-4x^2 \Rightarrow y-2=\pm\sqrt{64-4x^2} \Rightarrow y=2\pm\sqrt{64-4x^2}$$

B. Give an equation for a hyperbola with asymptotes $y=x$ and $y = -x$ and vertices (0,1) and (0, -1).

$$-x^2+y^2=1$$

$$-1)\cdot\left(-x^2+y^2=1 \Rightarrow y^2=x^2+1 \Rightarrow y=\pm\sqrt{x^2+1}\right)$$

C. Graph these two conic sections on the grid below.



D. Find the exact coordinates of all intersections of the two graphs.

$$(x,y)\in\left\{\left(-\frac{\sqrt{231}}{5},-\frac{16}{5}\right),\left(\frac{\sqrt{231}}{5},-\frac{16}{5}\right),\left(-\sqrt{15},4\right),\left(\sqrt{15},4\right)\right\}$$

$-x^2 + y^2 = 1$ $4E$ $-4x^2 + 4y^2 = 4$

$4x^2 + (y-2)^2 = 64$ E $4x^2 + (y-2)^2 = 64$

$4y^2 + (y-2)^2 = 68$

$4y^2 + y^2 - 4y + 4 = 68$

$5y^2 - 4y + -64 = 0$

$(5y+16)(y-4) = 0$

$(5y+16) = 0 \quad OR \quad (y-4) = 0$

$5y = -16 \quad OR \quad y = 4$

$y = -\frac{16}{5} = -3.2 \quad OR \quad y = 4$

$y = -\frac{16}{5} = -3.2$ Þ $-x^2 + \left(\frac{-16}{5}\right)^2 = 1$

$-x^2 + \left(\frac{256}{25}\right) = 1$

$-x^2 = -\frac{231}{25}$

$x^2 = \frac{231}{25}$

$x = \pm\sqrt{\frac{231}{25}}$

$x = \pm\frac{\sqrt{231}}{5}$ B ± 3.03973683071

$y = 4$ Þ $-x^2 + (4)^2 = 1$

$-x^2 + 16 = 1$

$-x^2 = -15$

$x^2 = 15$

$x = \pm\sqrt{15}$ B ± 3.87298334621

$(x,y) \in \left\{\left(-\frac{\sqrt{231}}{5}, -\frac{16}{5}\right), \left(\frac{\sqrt{231}}{5}, -\frac{16}{5}\right), \left(-\sqrt{15}, 4\right), \left(\sqrt{15}, 4\right)\right\}$