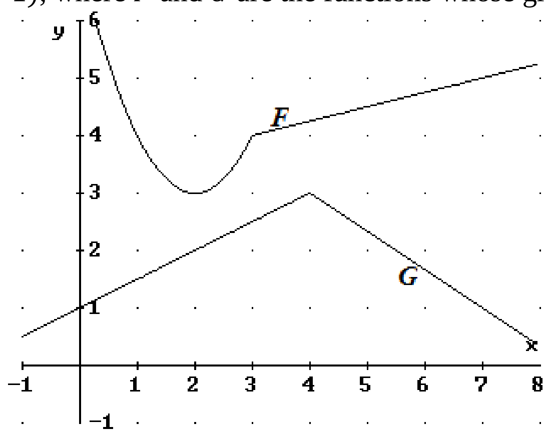
Arkansas Council of Teachers of Mathematics
2013 State Contest
Calculus Exam

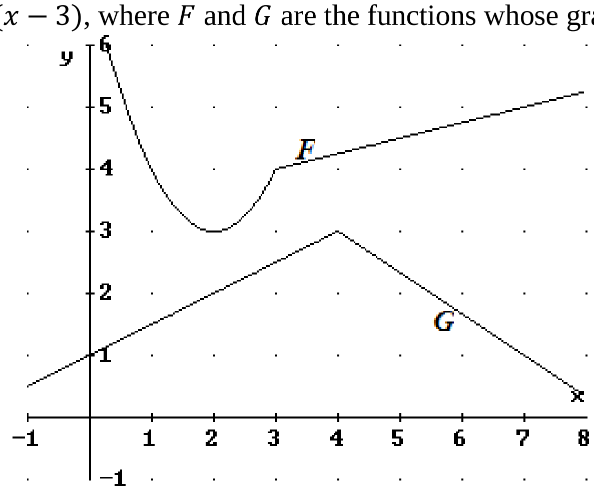
In each of the following choose the BEST answer and shade the corresponding letter on the Scantron Sheet. Answer all 24 multiple choice questions before attempting the tie-breaker questions. The tie-breaker questions at the end are to be used to resolve any ties between 1st, 2nd, and/or 3rd place. Be sure that your name is printed on each of the tiebreaker pages. The figures are not necessarily drawn to scale. You may use a scientific graphing calculator such as a TI-84. Good Luck!

1. Let $H(x) = F(x)G(3x+1)$, where F and G are the functions whose graphs are shown.



Find $H'(2)$.
A. -6
B. 0
C. 2
D. 3
E. $-\frac{2}{3}$

2. Let $K(x) = F(x) - G(x-3)$, where F and G are the functions whose graphs are shown.



Find $\int_3^7 K(x)\,dx$.
A. 26
B. 16
C. 10
D. 24
E. 12

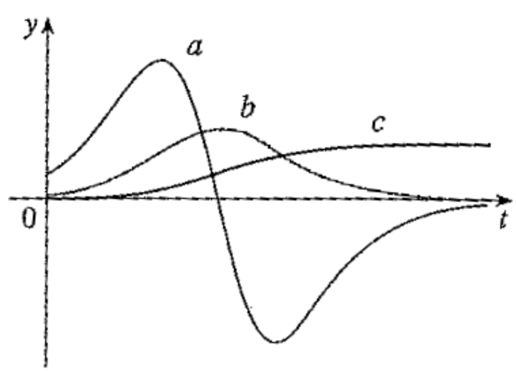
3. Suppose $f(1) = 7$ and $f'(x) \leq 3$ for all values of x. How large can $f(3)$ possibly be?
   A. 10
   B. 4
   C. 21
   D. 13
   E. 11

4. $\int_a^b f'(x)dx =$
   A. $f(b) - f(a)$
   B. $f'(b) - f'(a)$
   C. $f(a) - f(b)$
   D. $f'(a) - f'(b)$
   E. $f'(a) + f'(b)$

5. A particle moves along the curve $y = \sqrt{1 + x^3}$. As it reaches the point $(2,3)$, the y-coordinate is increasing at a rate of $4\ cm/s$. How fast is the x-coordinate of the point changing at that instant?

   A. $2\ cm/s$
   B. $6\ cm/s$
   C. $3\ cm/s$
   D. $24\ cm/s$
   E. $8\ cm/s$

6. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity function of the car, and one is the acceleration function of the car.



   Determine the valid statement.

   A. a is the position function.
   B. c is the velocity function.
   C. c is the acceleration function.
   D. b is the position function.
   E. b is the velocity function.

7. A canister is dropped from a helicopter at a height of 490 m. Its parachute does not open, but it is designed to withstand an impact velocity of 100 m/s. If we use 9.8 m/s^2 for the acceleration due to gravity and ignore wind resistance, then how fast is it going when it hits the ground?

   A. 0 m/s
   B. 98 m/s
   C. 9.8 m/s
   D. 980 m/s
   E. 10 m/s

8. A trough is 12 feet long and its cross-section has the shape of an isosceles triangle that is 3 feet at the top and has a height of 1 foot. If the trough is being filled with water at a rate of 12 ft^3/min, then how fast is the water level rising when the water is 6 inches deep?

   A. $3\, ft/\min$
   B. $\frac{2}{3}\, ft/\min$
   C. $6\, ft/\min$
   D. $\frac{1}{6}\, ft/\min$
   E. $\frac{2}{9}\, ft/\min$

9. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths a and b if the two sides of the rectangle lie along the legs.

   A. $\frac{ab}{2}$
   B. $\frac{2ab}{3}$
   C. $\frac{3ab}{2}$
   D. $\frac{3ab}{4}$
   E. $\frac{ab}{4}$

10. Suppose that we approximate a definite integral with a Left Riemann Sum. Which of the following statements is true?
    A. The estimate is lower than the actual integral if the function is increasing on the interval.
    B. The estimate is lower than the actual integral if the function is decreasing on the interval.
    C. The estimate is lower than the actual integral if the function is concave up on the interval.
    D. The estimate is lower than the actual integral if the function is concave down on the interval.
    E. Each of the other answers is incorrect.

11. Evaluate the limit by recognizing the expression as a limit of a Riemann sum for a function defined on [0,1] and applying the Fundamental Theorem of Calculus.

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\sqrt{\frac{3}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)$$

A. 1
B. $\frac{5}{9}$
C. $\frac{1}{2}$
D. 2
E. $\frac{2}{3}$

12. Suppose that we approximate a definite integral with the Trapezoidal Rule. Which of the following statements is true?
A. The estimate is lower than the actual integral if the function is increasing on the interval.
B. The estimate is lower than the actual integral if the function is decreasing on the interval.
C. The estimate is lower than the actual integral if the function is concave up on the interval.
D. The estimate is lower than the actual integral if the function is concave down on the interval.
E. Each of the other answers is incorrect.

13. Approximate $\int_1^9 x^5 dx$ using the Trapezoidal Rule with 4 intervals. Round your answer to the nearest whole number.
A. 40,352
B. 49,700
C. 88,573
D. 99,400
E. Each of the other answers is incorrect.

14. Evaluate the following: $\int 3x\sin\left(x^2\right)dx$.
A. $-3x\cos\left(x^2\right)+C$
B. $-\frac{3}{2}x^2\cos\left(x^2\right)+C$
C. $-\frac{3}{2}x^2\cos\left(\frac{1}{3}x^3\right)+C$
D. $-\frac{3}{2}\cos\left(x^2\right)+C$
E. Each of the other answers is incorrect.

15. $\lim_{x\to 4}\left(\frac{3x^2-3x-36}{x-4}\right)=21$. By the definition of a limit, there is a positive real number δ such that $\left|\left(\frac{3x^2-3x-36}{x-4}\right)-21\right|<0.09$ if $0<|x-4|<\delta$. The largest valid value of δ is

A. 0.01
B. 0.03
C. 0.3
D. 0.4
E. 0.07

16. Compute $\frac{d}{dx}\left(e^{\tan\left(3x^2\right)}\right)$

A. $e^{\tan(6x)}$
B. $e^{\tan\left(3x^2\right)}\sec^2(6x)$
C. $6xe^{\sec^2\left(3x^2\right)}$
D. $6xe^{\tan\left(3x^2\right)}\sec^2\left(3x^2\right)$
E. Each of the other answers is incorrect.

17. $\lim_{x\to 0}\left(\frac{4x}{\sin(3x)}\right)$

A. $\frac{4}{3}$
B. $\frac{3}{4}$
C. 1
D. 0
E. Each of the other answers is incorrect.

18. Find and simplify a formula for the difference quotient where $f(x)=x^3$.

A. $3x^2$
B. $3x^2+\Delta x$
C. $3x^2+\Delta x^2$
D. $3x^2+3x\Delta x+\Delta x^2$
E. Each of the other answers is incorrect.

19. Let $f(x)=\cos(2x)$. $f^{(16)}(0)=$

A. 0
B. 1
C. -1
D. -65,536
E. 65,536

20. The probability of an event $x \in [a, b]$ for a continuous probability distribution is the area between the graph of the probability density function (pdf) for the distribution and the x-axis over that interval. The wait time for starting service at a checkout line has probability distribution

$$pdf(x) = \begin{cases} 0.5e^{-0.5x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(x is measured in minutes.) What is the probability that the wait time will be between 1 and 3 minutes?
A. 0.15
B. 0.23
C. 0.38
D. 0.42
E. 0.53

21. The mean of a continuous probability distribution is $\mu = E[x] = \int_{-\infty}^{\infty} x \, pdf(x)\, dx$. What is the mean wait time for the distribution from the previous question?
A. 2.0 minutes
B. 4.0 minutes
C. 5.2 minutes
D. 1.4 minutes
E. Each of the other answers is incorrect.

For questions 22-24 let R be the region bounded by the curves $y = \sin\left(\frac{\pi}{4}x\right)$ and $y = \frac{1}{4}x^2$.

22. Approximate the area of the region R. Round your answer to four decimal places.
A. 0.2732
B. 0.3333
C. 0.6066
D. 1.2252
E. Each of the other answers is incorrect.

23. Approximate the volume of the solid formed by revolving the region R about the <u>x-axis</u>. Round your answer to four decimal places.
A. 1.1556
B. 1.8850
C. 3.8108
D. 3.9027
E. Each of the other answers is incorrect.

24. Approximate the volume of the solid formed by revolving the region R about the <u>y-axis</u>. Round your answer to four decimal places.
A. 1.1556
B. 1.8850
C. 3.8108
D. 3.9027
E. Each of the other answers is incorrect.

Tie-Breaker Questions

Name________________________________ **School___________________________________**

[Please Print] **[Please Print]**

In each of the following you must show supporting work for your answers to receive credit. The questions will be used in the order given to resolve ties for 1st, 2nd, and/or 3rd place. Be sure that your name is printed on each of the tiebreaker pages.

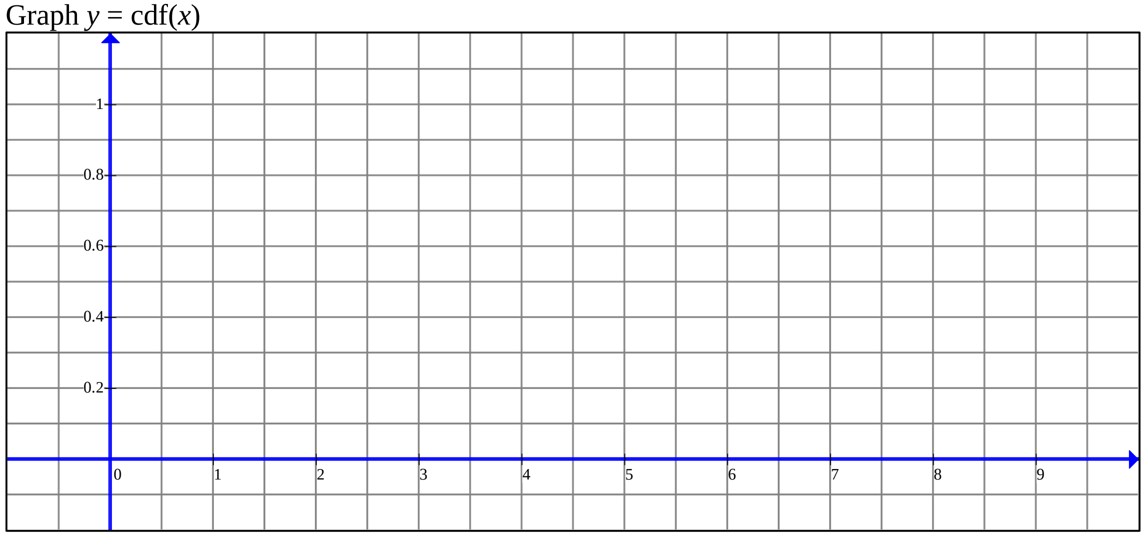
1. For a continuous probability distribution the probability that the random variable *x* is in the interval [*a*, *b*] is the area bounded by the *x*-axis, $x = a$, $x = b$ and the graph of the probability density function (pdf). The cumulative density function (cdf) for the distribution gives as its output the cumulative probability, i.e. cdf(*k*) = $P(x \leq k)$.

For a particular probability distribution we have

$$pdf(x) = \begin{cases} -\frac{1}{8}(x-7) & x \in [3,7] \\ 0 & otherwise \end{cases}$$

- Find the formula for cdf(*x*).

- Graph $y = $ cdf(*x*)



Tie-Breaker Questions

Name_______________________________ **School___________________________________**
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2. State the Sum Rule for calculating the derivative of a sum of two differentiable functions *f* and *g*:

$\frac{d}{dx}\left(f(x)+g(x)\right) =$ ____________________________

Prove this result.

Tie-Breaker Questions

Name________________________________ **School____________________________________**

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3. The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$ the population is 300, and when $t = 2$ the population has increased to 500. Find the population when $t = 3$.

Answers:
1. A
2. C
3. D
4. A
5. A
6. E
7. B
8. B
9. E
10. A
11. E
12. D
13. D
14. D
15. B
16. D
17. A
18. D
19. E
20. C
21. A
22. C
23. B
24. D

Tie Breaker #1 Name **Key**

For a continuous probability distribution, the probability that the random variable x is in the interval $[a, b]$ is the area bounded by the x-axis, $x = a$, $x = b$ and the graph of the probability density function (pdf). The cumulative density function (cdf) for the distribution gives as its output the cumulative probability, i.e. cdf(k) = P($x \le k$).

For a particular probability distribution we have

$$pdf(x) = \begin{cases} -\frac{1}{8}(x-7) & x \in [3,7] \\ 0 & otherwise \end{cases}$$

- Find the formula for cdf(x).

We see from the explanation above that in general the cdf is an antiderivative of the pdf. Specifically,

$$cdf(x) = \int_{-\infty}^{x} pdf(t)\,dt .$$

Notice that for $x < 3$ $cdf(x) = \int_{-\infty}^{x} 0\,dt = 0$ and note that $\int_{7}^{x} pdf(x)\,dt = \int_{7}^{x} 0\,dt = 0$ for $x > 7$.

For $x \in [3, 7]$

$$\begin{aligned} cdf(x) &= \int_3^x -\tfrac{1}{8}(t-7)\,dt \\ &= -\tfrac{1}{8}\left(\tfrac{1}{2}t^2 - 7t\right)\Big]_3^x \\ &= -\tfrac{1}{16}\left(\left(t^2-14t\right)\Big]_3^x\right) \\ &= -\tfrac{1}{16}\left(\left(x^2-14x\right)-\left((3)^2-14(3)\right)\right) \\ &= -\tfrac{1}{16}\left(x^2-14x+33\right) \\ &= -\tfrac{1}{16}x^2+\tfrac{7}{8}x-\tfrac{33}{16} \\ &= -\tfrac{1}{16}(x-11)(x-3) \\ &= -\tfrac{1}{16}(x-7)^2+1 \end{aligned}$$

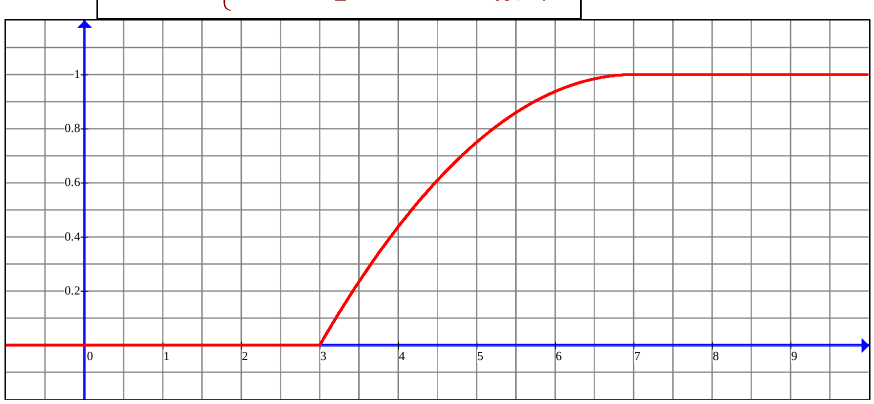
OR

$$\begin{aligned} cdf(x) &= \int_3^x -\tfrac{1}{8}(t-7)\,dt \qquad [u = t-7 \Rightarrow du = dt] \\ &= -\tfrac{1}{16}(t-7)^2\Big]_3^x \\ &= -\tfrac{1}{16}\left((t-7)^2\Big]_3^x\right) \\ &= -\tfrac{1}{16}\left((x-7)^2-(3-7)^2\right) \\ &= -\tfrac{1}{16}\left((x-7)^2-16\right) \\ &= -\tfrac{1}{16}(x-7)^2+1 \end{aligned}$$

(Any of the last four versions on the left are good in this interval and can be used for the middle piece below.)

$$\boxed{cdf(x) = \begin{cases} 0 & x < 3 \\ -\frac{1}{16}(x-11)(x-3) & x \in [3,7] \\ 1 & x > 7 \end{cases}}$$

Graph y = cdf(x)



Tie Breaker #2 Name **Key**

State the Sum Rule for calculating the derivative of a sum of two differentiable functions f and g:

$$\boxed{\tfrac{d}{dx}\left(f(x)+g(x)\right)=f'(x)+g'(x)}$$

Prove this result.

$\tfrac{d}{dx}\left(f(x)+g(x)\right)=\lim_{h\to 0}\dfrac{\left(f(x+h)+g(x+h)\right)-\left(f(x)+g(x)\right)}{h}$	Definition of the derivative.
$=\lim_{h\to 0}\dfrac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$	Distributive Property
$=\lim_{h\to 0}\dfrac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$	Commutative Property
$=\lim_{h\to 0}\dfrac{\left(f(x+h)-f(x)\right)+\left(g(x+h)-g(x)\right)}{h}$	Associative Property
$=\lim_{h\to 0}\left(\dfrac{f(x+h)-f(x)}{h}+\dfrac{g(x+h)-g(x)}{h}\right)$	Distributive Property
$=\lim_{h\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)+\lim_{h\to 0}\left(\dfrac{g(x+h)-g(x)}{h}\right)$	Limit of a sum is the sum of limits if they exist.
$=f'(x)+g'(x)$	Definition of a derivative.

Tie Breaker #3 Name ____________________________

The rate of change of the number of coyotes *N(t)* in a population is directly proportional to 650-*N(t)*, where *t* is the time in years. When *t* = 0 the population is 300, and when *t* = 2 the population has increased to 500. Find the population when *t* = 3.

$$\frac{dN}{dt} = k(650 - N) \Rightarrow \int \tfrac{1}{650-N}\,dN = k\int dt$$
$$-\ln(650 - N) = kt + C$$
$$\ln(650 - N) = -kt - C$$
$$650 - N = e^{-kt-C}$$
$$650 - N = e^{-C}e^{-kt}$$
$$650 - N = C_2e^{-kt}$$
$$N = 650 - C_2e^{-kt}$$
$$t = 0 \rightarrow N = 300$$
$$300 = 650 - C_2e^{-k0}$$
$$300 = 650 - C_2$$
$$C_2 = 350$$
$$N = 650 - 350e^{-kt}$$
$$t = 20 \rightarrow N = 500$$
$$500 = 650 - 350e^{-k2}$$
$$-150 = -350e^{-2k}$$
$$e^{-2k} = \frac{3}{7}$$
$$-2k = \ln\left(\tfrac{3}{7}\right)$$
$$k = -\tfrac{1}{2}\ln\left(\tfrac{3}{7}\right)$$
$$N = 650 - 350e^{\frac{1}{2}\ln\left(\frac{3}{7}\right)t} = 650 - 350\left(\tfrac{3}{7}\right)^{\frac{t}{2}}$$
$$N(3) = 650 - 350e^{\frac{1}{2}\ln\left(\frac{3}{7}\right)3} = 650 - 350\left(\tfrac{3}{7}\right)^{\frac{3}{2}} = 551.802$$

When *t* = 3 the population is 552.