Arkansas Council of Teachers of Mathematics
Algebra II State Contest 2016

Work the multiple choice questions first, selecting the single best response from those provided and entering it on your scantron form. You may write on this test and keep the portion with the multiple choice questions. After completing the multiple choice section work the open response tie breakers in the order presented. Tie breaker 2 will be used only if there is still a tie after the first tie breaker and similarly the third tie breaker will be used only if there is a tie after reading the second tie breaker. You will turn in all three tie breakers with your name on the pages regardless of what you have or have not written there.

1. $e^{\ln(a)+\ln(b)} =$

   a) $a+b$ b) a^b c) ab d) e^{ab} e) Each of the other answers is incorrect.

2. If a and b are positive real numbers and $a > b$, then $\left(\dfrac{a}{b}\right)^{-1}$ is

   a) less than -1.
   b) between -1 and 0.
   c) between 0 and 1.
   d) greater than 1.
   e) There is insufficient information to draw a conclusion.

3. A bacteria culture has an initial population of P_0 cells. The number P of cells doubles every three days. What will be the population P of cells 7 days after time 0?

   a) $4\sqrt{3}P_0$
   b) $4\sqrt[3]{2}P_0$
   c) $\dfrac{7}{3}P_0$
   d) $\dfrac{14}{3}P_0$
   e) Each of the other answers is incorrect.

4. $\dfrac{1}{5x^3\sqrt[4]{x}} =$

   a) $-5x^{\frac{13}{4}}$ b) $-5x^{-1}$ c) $-\dfrac{1}{5}x^{\frac{4}{13}}$ d) $\dfrac{1}{5}x^{-\frac{13}{4}}$ e) $\dfrac{1}{5}x^{-\frac{3}{4}}$

5. Solve: $\dfrac{32}{a} - 1 < -2$

   a) $a < -32$ b) $0 < a < 32$ c) $-32 < a < 0$ d) $a > -32$
   e) Each of the other answers is incorrect.

6. To the nearest degree, 3.141593 radians is equivalent to how many degrees?

   a) 90º b) 180º c) 360º d) 565º e) 1131º

7. The formula to convert a temperature in Celsius degrees to the equivalent temperature in Fahrenheit degrees is $F = \frac{9}{5}C + 32$. The inverse relation to convert a Fahrenheit measure to the equivalent Celsius temperature is

a) $C = \frac{9}{5}F + 32$
b) $C = \frac{5}{9}F - 32$
c) $F = \frac{5}{9}(C - 32)$
d) $C = \frac{5}{9}(F - 32)$
e) Each of the other answers is incorrect.

8. Let ε represent a small positive number. If $|(-2x+7)-3| < \varepsilon$, then

a) $|x-2| < \frac{\varepsilon}{2}$
b) $|x-2| < \frac{\varepsilon}{-2}$
c) $|x+5| < \frac{\varepsilon}{2}$
d) $|x| < \frac{\varepsilon - 10}{2}$
e) Each of the other answers is incorrect.

9. If $a^m = b$, then $\ln(b) =$

a) ma
b) $m\ln(a)$
c) $a\ln(m)$
d) $\ln(e^{ma})$
e) Each of the other answers is incorrect.

10. The *x*-intercepts of a quadratic function are $\left(2-\sqrt{3},0\right)$ and $\left(2+\sqrt{3},0\right)$. The *y*-intercept is $(0,-2)$. The quadratic function is

a) $f(x) = -3x^2 + 5$
b) $f(x) = -2x^2 + 8x - 11$
c) $f(x) = -3x^2 + 12x - 3$
d) $f(x) = x^2 - 4x + 1$
e) $f(x) = -2x^2 + 8x - 2$

11. The quadratic equation $3ax^2 + 2bx + c = 0$ has no real solutions for *x* if and only if

a) $b^2 - 4ac < 0$
b) $b^2 - 4ac > 0$
c) $b^2 - 3ac < 0$
d) $4b^2 + 12ac < 0$
e) It always has real solutions.

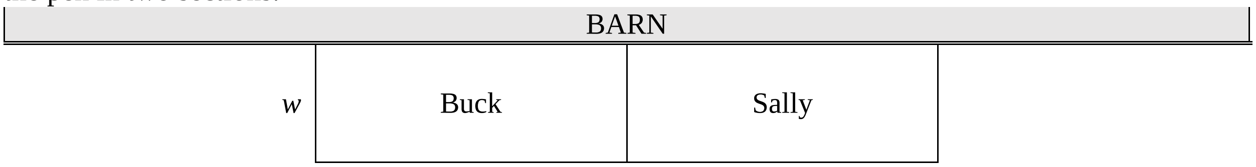
12. Over the last two years (from March 3, 2014 to February 29, 2016), the weekly price of a share of *Apple* stock can be approximated quite well by the quadratic function

$$S = -0.0141w^2 + 1.7748w + 64.977$$

where *w* is the number of weeks since March 3, 2014 and *S* is the price of a share of Apple stock. Determine the week *w* in which the formula estimates Apple reached its highest price.

a) 60
b) 63
c) 70
d) 74
e) Each of the other answers is incorrect.

Questions 13-14: Farmer John needs to build a pen for his dogs Buck and Sally. John plans to build a rectangular pen next to the barn which will serve as one boundary of the pen with a center fence dividing the pen in two sections.



The total area of the pen should be 50 square meters and the width of the pens is *w*.

13. The amount of fence *F* required for the pens is given by the function

a) $F = 3w + \frac{50}{w}$ b) $F = 3w + \frac{100}{w}$ c) $F = w(50 - 3w)$ d) $F = 50w$

e) Each of the other answers is incorrect.

14. To the nearest hundredth of a meter what is the minimum amount of fencing required?

a) 4.08 meters b) 208.33 meters c) 36.6 meters d) 24.49 meters

e) Each of the other answers is incorrect.

15. A parabola has a vertical axis of symmetry. The vertex of the parabola is at $(-1,4)$ and the *y*-intercept of the parabola is $(0,2)$. What are the *x*-intercepts of the parabola?

a) $\left(-1-\sqrt{2},0\right)$ *and* $\left(-1+\sqrt{2},0\right)$ b) $\left(2-2\sqrt{2},0\right)$ *and* $\left(2+2\sqrt{2},0\right)$

c) (-3,0) *and* (1, 0) d) $\left(-1-2\sqrt{2},0\right)$ *and* $\left(-1+2\sqrt{2},0\right)$

e) Each of the other answers is incorrect.

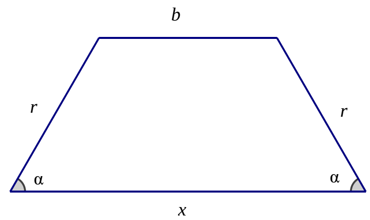
16. Asa boards a Ferris wheel at the Arkansas State Fair. *t* minutes after his ride begins, Asa's height *h* above the ground is given by the function $h(t) = 10 + 8\sin\left(3\pi t - \frac{\pi}{2}\right)$. If the ride lasts four minutes, how many complete revolutions will Asa go around the wheel?

a) 1.5 revolutions b) 2 revolutions c) 4 revolutions d) 6 revolutions e) 12 revolutions

17. Tim boards a smaller Ferris wheel at the Arkansas State Fair with his children. *t* minutes after his ride begins, Tim's height *h* above the ground is given by the function $h(t) = 3 + 2\sin\left(\frac{\pi}{2}t - \frac{\pi}{2}\right)$. At what time will Tim and his kids reach the top of the wheel the second time?

a) 10 minutes b) 6 minutes c) 4 minutes d) 2 minutes e)1 minute

18. We are given a general isosceles trapezoid with the shorter base having length b, the two legs having length r, and the smaller base angles having measure α. What is the length of the longer base x?



a) $2b$ b) $b + r$ c) $b+\frac{1}{2}r$ d) $b+2r\sin(\alpha)$ e) $b+2r\cos(\alpha)$

19. Bob's Pizza is best known for super large pizzas. After experimenting with several prices and the resulting sales, Bob has determined that the number q of super large pizzas Bob will sell on a Friday night is related to the price p per pizza by the relation $q = 200 - 10p$. This formula tells us

a) Some customers will pay up to \$200 for a pizza, but each \$1 increase in the price causes Bob to sell 10 fewer pizzas.
b) Some customers will pay up to \$200 for a pizza, but to gain one more customer Bob must decrease the price by \$10.
c) To gain one more customer, Bob must decrease the price by \$10, and Bob can give away 200 pizzas for free.
d) Bob can give away 200 pizzas for free, and each increase of \$1 in the price causes Bob to lose 10 customers.
e) Each of the other answers is incorrect.

20. In 2002, the number of people in Algebraville was 346. By 2010, the number of people in Algebraville was 547. If we assume the population is growing according to an exponential model, then the number of people P in Algebraville in 2016 should be

a) $P = 346\left(\frac{547}{346}\right)^{\frac{3}{4}}$ b) $P = 547\left(\frac{346}{547}\right)^{\frac{3}{4}}$ c) $P = 346\left(\frac{547}{346}\right)^{\frac{7}{4}}$ d) $P = 346\left(\frac{547}{346}\right)^{\frac{4}{3}}$

e) Each of the other answers is incorrect.

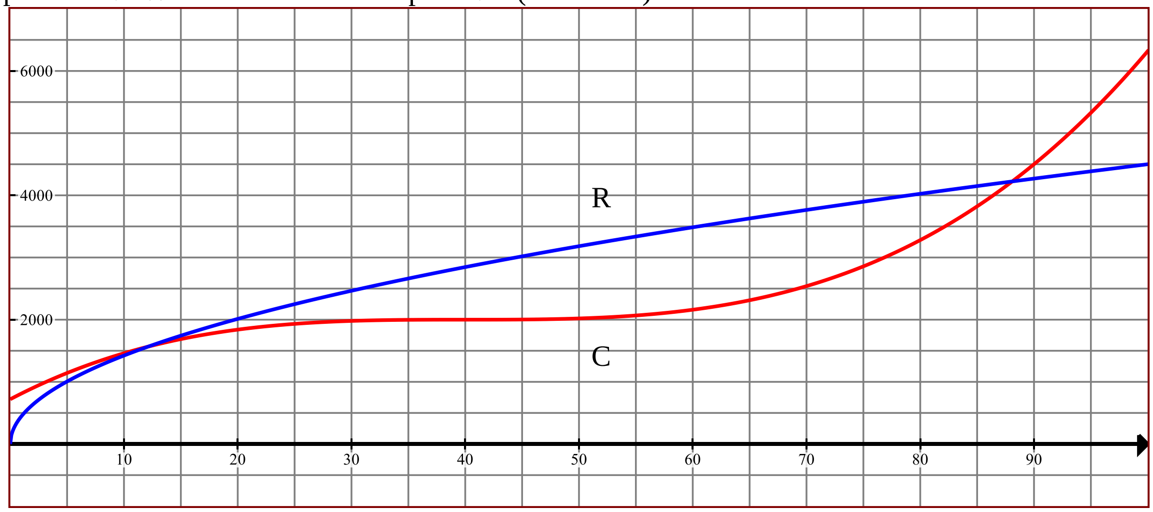
21. Consumers respond to higher prices by purchasing less, while suppliers react to higher prices by producing more. In Geometryville (you've all been there), the weekly quantity q of pizzas consumers will purchase at a price p can be modeled by the formula $q = 3000 - 150p$. The quantity q suppliers (pizza makers) will produce at a price p can be modeled by the formula $q = 200p - 600$. At what price p will the number of pizzas produced by suppliers match the number of pizzas consumers will purchase?

a) \$10.29 b) \$11.50 c) \$15.63 d) \$20 e) \$1457.14

22. The graph of a quadratic polynomial function $f(x) = ax^2 + bx + c$ passes through the points (-1, 5), (2, -3), and (3, 7). What is the value of c?

a) 4 b) 2 c) 0 d) 1 e) -4

23. The following graphs give the cost and revenue (in dollars) as functions of the number of items produced. What is the maximum profit? (P = R – C)



a) \$1800 b) \$12 c) \$1000 d) \$1330 e) \$62

24. Consider the solution set of the equation $x^3 = 1$ in the complex numbers. The **product** of the solutions is _____.

a) 1 b) 0 c) $\frac{2}{3}$ d) $\frac{7}{4}$ e) Each of the other answers is incorrect.

25. There is an urn with 4 black balls and 6 red balls. You randomly select one ball and then without replacing it randomly select another ball. What is the probability that the two balls are the same color?

a) $\frac{7}{15}$ b) $\frac{8}{15}$ c) $\frac{1}{3}$ d) $\frac{13}{15}$ e) $\frac{1}{5}$

ACTM State Algebra II Competition 2016
Tie Breaker 1

Name ______________________________

Suppose that we have a power function

$$f(x) = x^{\frac{p}{q}}$$

where the exponent is a reduced fraction (so p and q are integers with a greatest common divisor of 1 and q is not zero). The quadrants in which the graph of $y = f(x)$ lies are determined by whether p and q are even or odd. Complete the table below by determining the quadrant(s) of the graph for each combination of p and q even or odd. Justify your answer for each case.

Case for p and q	Graph of $f(x) = x^{\frac{p}{q}}$ lies in quadrant(s) (Circle correct quadrant(s))	Justify your answer.
p is odd and q is odd	I II III IV	
p is odd and q is even	I II III IV	
p is even and q is odd	I II III IV	

ACTM State Algebra II Competition 2016

Tie Breaker 2

Name ______________________________

Find the values of A and B for which the following equation is true for all values of x where the left side is defined. Show your work in arriving at your answers.

$$\frac{13x+12}{(2x+3)(x-1)} = \frac{A}{2x+3} + \frac{B}{x-1}$$

A = ___, B = ___

ACTM State Algebra II Competition 2016
Tie Breaker 3

Name ______________________________

A. Given points (x_1,y_1,z_1) and (x_2,y_2,z_2) in a three-dimensional coordinate system, what is a formula for the distance between the two points?

B. Use the formula to find the distance between (1, 2, -3) and (-4, 6, -2).

C. Explain/Prove why this formula works.

Answers

1	C
2	C
3	B
4	D
5	C
6	B
7	D
8	A
9	B
10	E
11	C
12	B
13	A
14	D
15	A
16	D
17	B
18	E
19	D
20	C
21	A
22	E
23	D
24	A
25	A

ACTM State Algebra II Competition 2016
Tie Breaker 1

Name **Solution Key**

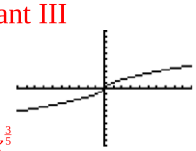
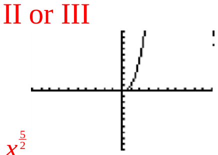
Suppose that we have a power function

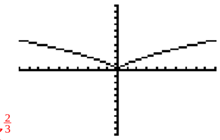
$$f\left(x\right) = x^{\frac{p}{q}}$$

where the exponent is a reduced fraction (so p and q are integers with a greatest common divisor of 1 and q is not zero). The quadrants in which the graph of $y = f(x)$ lies are determined by whether p and q are even or odd. Complete the table below by determining the quadrant(s) of the graph for each combination of p and q even or odd. Justify your answer for each case.

First note that a positive number x to *any* power is a positive number $f(x)$, so all of these graphs have a portion in the first quadrant, and they never enter the fourth quadrant. Now we just have to consider what happens when x is a negative number. Note other ways to write the function are

$$f(x) = x^{\frac{p}{q}} = \left(x^p\right)^{\frac{1}{q}} = \sqrt[q]{x^p}$$

Case for p and q	Graph of $f(x) = x^{\frac{p}{q}}$ lies in quadrant(s) (Circle correct quadrant(s))	Justify your answer.
p is odd and q is odd	$\boxed{\text{I}}$ II $\boxed{\text{III}}$ IV	A negative to an odd power is negative and then an odd root of this negative is also negative. E.g. $(-1)^{\frac{5}{3}} = \sqrt[3]{(-1)^5} = \sqrt[3]{-1} = -1$ Negative input gives negative output → Quadrant III  Graph of $f(x) = x^{\frac{3}{5}}$
p is odd and q is even	$\boxed{\text{I}}$ II III IV	A negative to an odd power is negative and then an even root of this negative is non-real. E.g. $(-1)^{\frac{5}{2}} = \sqrt{(-1)^5} = \sqrt{-1} = i \notin \mathbb{R}$ Negative input makes the graph undefined → No Quadrant II or III  Graph of $f(x) = x^{\frac{5}{2}}$

p is even and q is odd	$\boxed{\text{I}}$ $\boxed{\text{II}}$ III IV	A negative to an even power is positive and then an odd root of this positive is positive. E.g. $\left(-1\right)^{\frac{2}{3}}=\sqrt[3]{\left(-1\right)^{2}}=\sqrt[3]{1}=1$ Negative input gives positive output → Quadrant II  Graph of $f\left(x\right)=x^{\frac{2}{3}}$

ACTM State Algebra II Competition 2016
Tie Breaker 2

Name **Solution Key**

Find the values of A and B for which the following equation is true for all values of x where the left side is defined. Show your work in arriving at your answers.

$$\frac{13x+12}{(2x+3)(x-1)}=\frac{A}{2x+3}+\frac{B}{x-1}$$

A = **3**, B = **5**

For all x:

$$\frac{13x+12}{(2x+3)(x-1)}=\frac{A}{2x+3}+\frac{B}{x-1}$$

$$\frac{13x+12}{(2x+3)(x-1)}=\frac{A(x-1)}{(2x+3)(x-1)}+\frac{(2x+3)B}{(2x+3)(x-1)}$$

$$\frac{13x+12}{(2x+3)(x-1)}=\frac{A(x-1)+B(2x+3)}{(2x+3)(x-1)}$$

$$\underline{\underline{13x+12=A(x-1)+B(2x+3)}}$$

$\underline{Let \quad x=1}$

$$13(1)+12=A((1)-1)+B(2(1)+3)$$

$$25=0+5B$$

$$\boxed{5=B}$$

$$13x+12=A(x-1)+5(2x+3)$$

$\underline{Let \quad x=0}$

$$13(0)+12=A((0)-1)+5(2(0)+3)$$

$$12=-A+15$$

$$-3=-A$$

$$\boxed{3=A}$$

OR

$$\underline{\underline{13x+12=A(x-1)+B(2x+3)}}$$

$$13x+12=Ax-A+2Bx+3B$$

$$13x+12=(A+2B)x+(-A+3B)$$

$$A+2B=13$$

$$-A+3B=12$$

$$5B=25$$

$$\boxed{B=5}$$

$$A+2(5)=13$$

$$A+10=13$$

$$\boxed{A=3}$$

ACTM State Algebra II Competition 2016

Tie Breaker 3

Name **Solution Key**

A. Given points (x_1,y_1,z_1) and (x_2,y_2,z_2) in a three-dimensional coordinate system what is a formula for the distance between the two points.

$$d=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

B. Use the formula to find the distance between (1, 2, -3) and (-4, 6, -2).

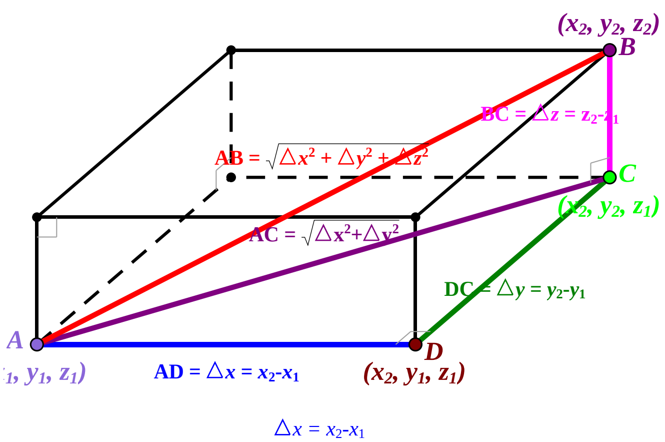
$$d=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{({}^{-}4-1)^2+(6-2)^2+(-2-{}^{-}3)^2}=\sqrt{({}^{-}5)^2+(4)^2+(1)^2}$$
$$=\sqrt{25+16+1}=\boxed{\sqrt{42}}\approx 6.480740698$$

C. Explain/Prove why this formula works.

Three Dimensional Euclidean Distance Formula

$A=(x_1, y_1, z_1)$ & $B=(x_2, y_2, z_2)$

$AB=\sqrt{\triangle x^2+\triangle y^2+\triangle z^2}$



We will apply the Pythagorean Theorem twice to derive a three dimensional Euclidean Distance Formula.

Suppose that we want to find the distance between two points in three dimensional space: $A=(x_1, y_1, z_1)$ and $B=(x_2, y_2, z_2)$. First consider the point $C=(x_2, y_2, z_1)$ which is directly above or below point B and is in the same horizontal plane as point A. Next we consider the point $D=(x_2, y_1, z_1)$.

Note that $\triangle ADC$ is a right triangle in the horizontal plane $z=z_1$ with a right angle at D. Notice that the lengths of the two legs of this right triangle are $AD=\triangle x=x_2-x_1$ and $DC=\triangle y=y_2-y_1$. By the Pythagoren Theorem, $AC^2=\triangle x^2+\triangle y^2$, so the length of the hypotenuse (the diagonal on the bottom face of the pictured prism) is $AC=\sqrt{\triangle x^2+\triangle y^2}$.

Now consider $\triangle ACB$ which is also a right triangle with right angle at point C. $BC=\triangle z=z_2-z_1$. Notice that while AC was the hypotenuse of $\triangle ADC$ it is one of the legs of $\triangle ACB$. We again apply the Pythagoren Theorem to see that $AB=\sqrt{AC^2+BC^2}=\sqrt{\triangle x^2+\triangle y^2+\triangle z^2}$.