Arkansas Council of Teachers of Mathematics
Algebra II State Contest 2015

Work the multiple choice questions first, selecting the single best response from those provided and entering it on your scantron form. You may write on this test and keep the portion with the multiple choice questions. After completing the multiple choice section work the open response tie breakers in the order presented. Tie breaker 2 will be used only if there is still a tie after the first tie breaker and similarly the third tie breaker will be used only if there is a tie after reading the second tie breaker. You will turn in all three tie breakers with your name on the pages regardless of what you have or have not written there.

1. $\log_{\pi}(1) =$

   A. 0
   B. $\frac{1}{\pi}$
   C. 1
   D. π
   E. None of the other answers is correct.

2. $\log_3\left(\frac{1}{\sqrt[7]{243}}\right) =$

   A. -0.71
   B. $-\frac{5}{7}$
   C. $-\frac{7}{5}$
   D. $\frac{7}{5}$
   E. None of the other answers is correct.

3. One zero of a quadratic function $P(x)$ with real coefficients is $x = 3 + 2i$. The y-intercept of the function is negative. Which of the following could be this function?

   A. $P(x) = (x - 2 - 3i)^2$
   B. $P(x) = -(x - 2 - 3i)^2$
   C. $P(x) = (x - 3 + 2i)(x + 3 - 2i)$
   D. $P(x) = -(x - 3 - 2i)(x - 3 + 2i)$
   E. None of the other answers is correct.

4. If $x < 2$, then $|x-2| =$

A. $x-2$
B. $x+2$
C. $2-x$
D. $-2-x$
E. None of the other answers is correct.

5. John's Pizza normally sells 100 pizzas a night at a price per pizza of \$10. When John puts the pizzas on sale for \$6 each, he sells 150 pizzas in a night. Assume the relation between the price p per pizza and the quantity q of pizzas sold is linear. What is the nightly <u>revenue</u> R (income not number of pizzas) as a function of the price p?

A. $R = 225 - 12.5p$
B. $R = 225p - 12.5p^2$
C. $R = 18p - 0.08p^2$
D. $R = 18p - 0.08pq$
E. None of the other answers is correct.

6. A parabola passes through the points $(-1,5)$, $(2,-3)$, and $(3,7)$. By substituting into $f(x) = ax^2 + bx + c$, Loretta uses the three points to obtain a system of equations for the coefficients a, b, and c. What is Loretta's system?

A. $\begin{aligned} -a - b + c &= 5 \\ 2a + 2b + c &= -3 \\ 3a + 3b + c &= 7 \end{aligned}$

B. $\begin{aligned} a - b + c &= 5 \\ 4a + 2b + c &= -3 \\ 9a + 3b + c &= 7 \end{aligned}$

C. $\begin{aligned} 5a + 5b + c &= -1 \\ -3a - 3b + c &= 2 \\ 7a + 7b + c &= 3 \end{aligned}$

D. $\begin{aligned} 25a + 5b + c &= -1 \\ 9a - 3b + c &= 2 \\ 49a + 7b + c &= 3 \end{aligned}$

E. None of the other answers is correct.

7. Let A be the area of a circle and let C be the circumference C of the circle. Formulas for these measurements are typically given as functions of the radius r. Write a formula for the area A as a function of the circumference C.

A. $A(C)=\dfrac{C^2}{4\pi}$
B. $A(C)=2\pi C^2$
C. $A(C)=4\pi^3C^2$
D. $A(C)=\dfrac{C^2}{4}$
E. None of the other answers is correct.

8. The period T of a simple pendulum is given by the formula $T=2\pi\sqrt{\dfrac{L}{g}}$ where L is the length of the pendulum and g is the acceleration due to gravity. Solving this formula for g yields

A. $g=\dfrac{2\pi L}{T^2}$
B. $g=L\sqrt{\dfrac{2\pi}{T}}$
C. $g=\dfrac{4\pi^2 L}{T^2}$
D. $g=\dfrac{T^2}{4\pi^2 L}$
E. None of the other answers is correct.

9. Let $e^p=a$, $e^q=b$, and $e^r=c$, then $\ln\left(\dfrac{a\sqrt{b}}{c}\right)=$

A. $\dfrac{p\sqrt{q}}{r}$
B. $\dfrac{p-2q}{r}$
C. $p-2q-r$
D. $p+\dfrac{q}{2}-r$
E. None of the other answers is correct.

10. The three most used temperatures scales are *Fahrenheit*, *Celsius*, and *Kelvin*. Water freezes at 32º*F*, 0º*C*, and 273*K*. Water boils at 212º*F*, 100º*C*, and 373*K*. The linear function giving the Fahrenheit temperature *F* equivalent to a Kelvin temperature *K* is

A. $F = \frac{5}{9}K - 119.67$
B. $F = \frac{5}{9}K + 255.22$
C. $F = \frac{9}{5}K - 241$
D. $F = \frac{9}{5}K - 459.4$
E. None of the other answers is correct.

11. On a large planet with no atmosphere, an astronaut shoots an arrow from a crossbow. The height *h* (in meters) of the arrow above the ground *t* seconds after being shot is given by the function

$$h(t) = -10t^2 + 50t + 2.$$

How long after it is shot is the arrow 42 meters high?

A. 2.5 seconds
B. 1 second.
C. 1 second and 4 seconds.
D. 5.04 seconds.
E. None of the other answers is correct.

12. $f(x) = 5.7\sin(2.3x + 3.7) + 11.3$. The *range* of this function is

A. $[-1.61, 1.12]$
B. $[-5.7, 11.3]$
C. $[5.6, 17.0]$
D. $[-5.7, 5.7]$
E. None of the other answers is correct.

13. A mass hangs at rest at the end of a spring, 10 cm above a table. The mass is pulled down and released, causing the mass to bob up and down. The vertical position y of the mass above the table after t seconds is given by

$$y = -3\sin\left(2\pi t + \frac{\pi}{2}\right) + 10$$

When does the mass pass the middle height of 10 cm for the 45$^{\text{th}}$ time?

A. $-\frac{1}{4}$ seconds
B. $21\frac{3}{4}$ seconds
C. $22\frac{1}{4}$ seconds
D. $44\frac{3}{4}$ seconds
E. None of the other answers is correct.

14. $f(x) = \cos\left(x^2\right)$. This function completes one cycle (between successive relative maxima) over which interval of x values?

A. $10\pi \le x \le 12\pi$
B. $100\pi^2 \le x \le 144\pi^2$
C. $\sqrt{10\pi} \le x \le 2\sqrt{3\pi}$
D. $10\pi - \sqrt{\pi} \le x \le 10\pi + \sqrt{\pi}$
E. None of the other answers is correct.

15. The function $f(x) = 2\sec\left(2x + \frac{\pi}{3}\right) + \sqrt{3}$ has a vertical asymptote at

A. $x = -\frac{23}{12}\pi$
B. $x = -\frac{11}{6}\pi$
C. $x = -\frac{1}{12}\pi$
D. $x = \frac{5}{12}\pi$
E. None of the other answers is correct.

16. Household electricity (from the electrical outlets in your bedroom wall) is called *alternating current* because the voltage alternates between positive and negative values. In fact, the voltage V is given by a sine function of the time t. For standard household current, the amplitude of the voltage is $120\sqrt{2}$ volts and the voltage completes 60 full cycles every second. Which of the following could be the voltage function $V = V(t)$?

A. $V = 60\sin\left(\dfrac{1}{120\sqrt{2}}t\right)$
B. $V = 60\sin\left(120\sqrt{2}t\right)$
C. $V = 120\sqrt{2}\sin\left(\dfrac{\pi}{30}t\right)$
D. $V = 120\sqrt{2}\sin\left(120\pi t\right)$
E. None of the other answers is correct.

17. In standard position, the terminal ray of angle θ lies in quadrant II and $\sin(\theta) = \dfrac{u}{3}$. $\tan(\theta) =$

A. $\dfrac{u}{u-3}$
B. $\dfrac{3}{3-u}$
C. $-\dfrac{u}{\sqrt{9-u^2}}$
D. $\dfrac{u}{\sqrt{u^2-9}}$
E. $\dfrac{u}{\sqrt{9-u^2}}$

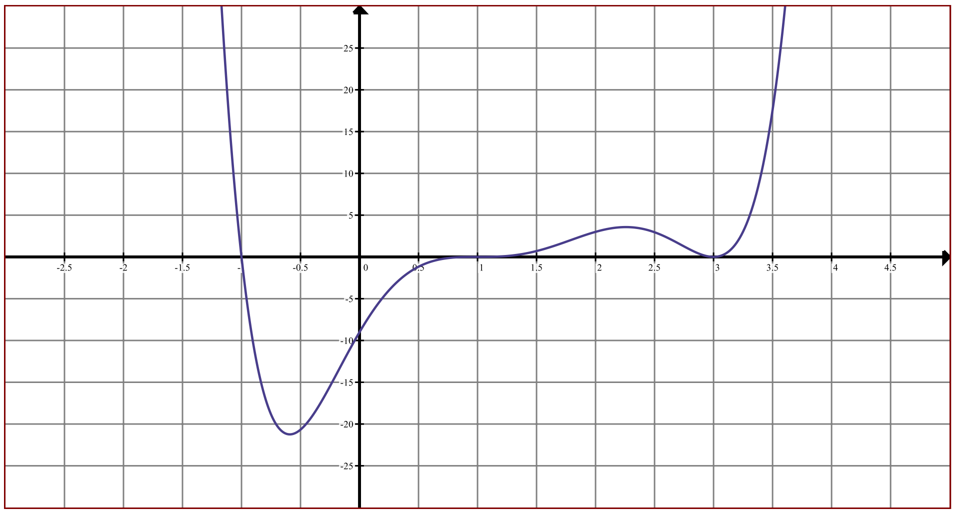
18. Describe the graph of $3x^2 - 4x - 5y^2 + 2y - 7 = 0$.

A. Circle
B. Ellipse
C. Parabola
D. Hyperbola
E. Each of the other answers is incorrect.

19. The annual amount of rainfall in a particular region is normally distributed with a mean of 16 inches and a standard deviation of 3 inches. What is the probability that the region will get between 14 and 17 inches of rainfall this year?

A. 50%
B. 29%
C. 42%
D. 31%
E. 38%

Questions 20-21. In the following graph of a polynomial function all x and y intercepts, relative extrema, inflection points, and end behavior are visible.



20. Which of the following is true about the function?

A. The degree is even and the leading coefficient is positive.
B. The degree is even and the leading coefficient is negative.
C. The degree is odd and the leading coefficient is positive.
D. The degree is odd and the leading coefficient is positive.
E. None of the conditions above can be guaranteed.

21. Which of the following is a possible value of the degree of the polynomial n?

A. $n = 3$
B. $n = 4$
C. $n = 5$
D. $n = 6$
E. $n = 7$

22. $\dfrac{b^{2k+2}b^{k-3}}{\left(b^{k+1}\right)^2} =$

A. $b^{\frac{3k-1}{(k+1)^2}}$
B. $b^{\frac{3k-1}{2k+2}}$
C. b^{k+1}
D. b^{k-3}
E. None of the other answers is correct.

23. In the American casino game of roulette there are 18 black slots, 18 red slots, and two green slots. A ball is placed in the spinning wheel and is equally likely to land in any of the slots. A player betting on black will double their money if the ball lands in a black slot and will lose it all if it lands anywhere else. On average, approximately how much money does the casino expect to make on each dollar bet on black in roulette?

A. 0
B. \$0.05
C. \$0.10
D. \$1
E. It cannot be determined.

24. Consider the power function $f(x) = x^{\frac{p}{q}}$ where gcd(p, q) = 1. Suppose that p is even and q is odd then we know that

A. The graph of f is in quadrants I and II and has origin symmetry.
B. The graph of f is in quadrants I and II and has y-axis symmetry.
C. The graph of f is in quadrants I and III and has y-axis symmetry.
D. The graph of f is in quadrants I and III and has origin symmetry.
E. The graph of f is in quadrant I only.

25. There is an urn containing 4 red balls, 6 black balls, and 5 green balls. Three balls are drawn out without replacement. What is the probability that one of each color is drawn?

A. $\frac{24}{91}$
B. $\frac{4}{91}$
C. $\frac{16}{75}$
D. $\frac{8}{225}$
E. None of the other answers is correct.

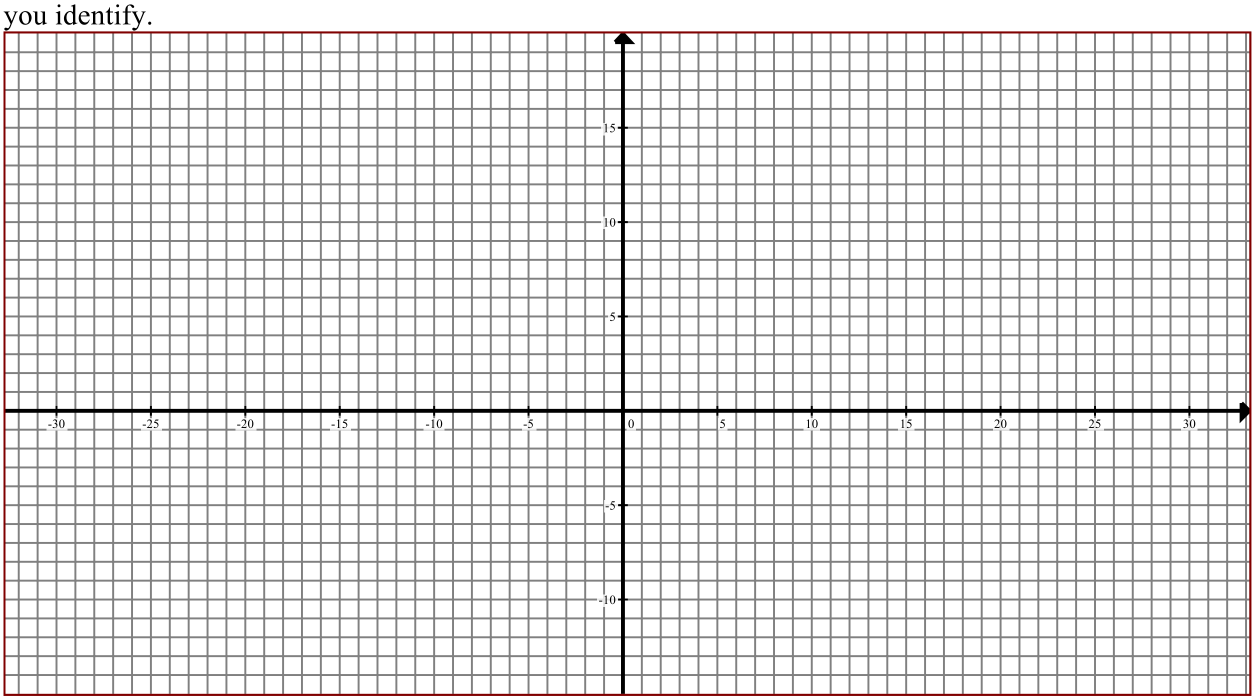
ACTM State Algebra II Competition 2015
Tie Breaker 1

Name ____________________________

Clearly and carefully graph the function

$$f\left(x\right)=\frac{x^3+x^2-8x-12}{2x^2-10x+12}$$

on the grid below and then comment on important features of the graph including such things as domain and range, and any intercepts, extrema, holes, and asymptotes. Justify your conclusions for each feature you identify.



ACTM State Algebra II Competition 2015
Tie Breaker 2

Name ____

The first two terms of a sequence are 4 and 7 in that order. When we find the differences between successive terms this produces a sequence of first differences. When we find the difference between successive terms of the sequence of first differences we obtain a sequence of second differences. The sequence of second differences is a constant sequence with value 4.

A. Write out the first 8 output values of the original sequence in order.

B. Write out an explicit non-recursive formula for the k^{th} term of the original sequence.

ACTM Regional Algebra II Competition 2015
Tie Breaker 3

Name ____

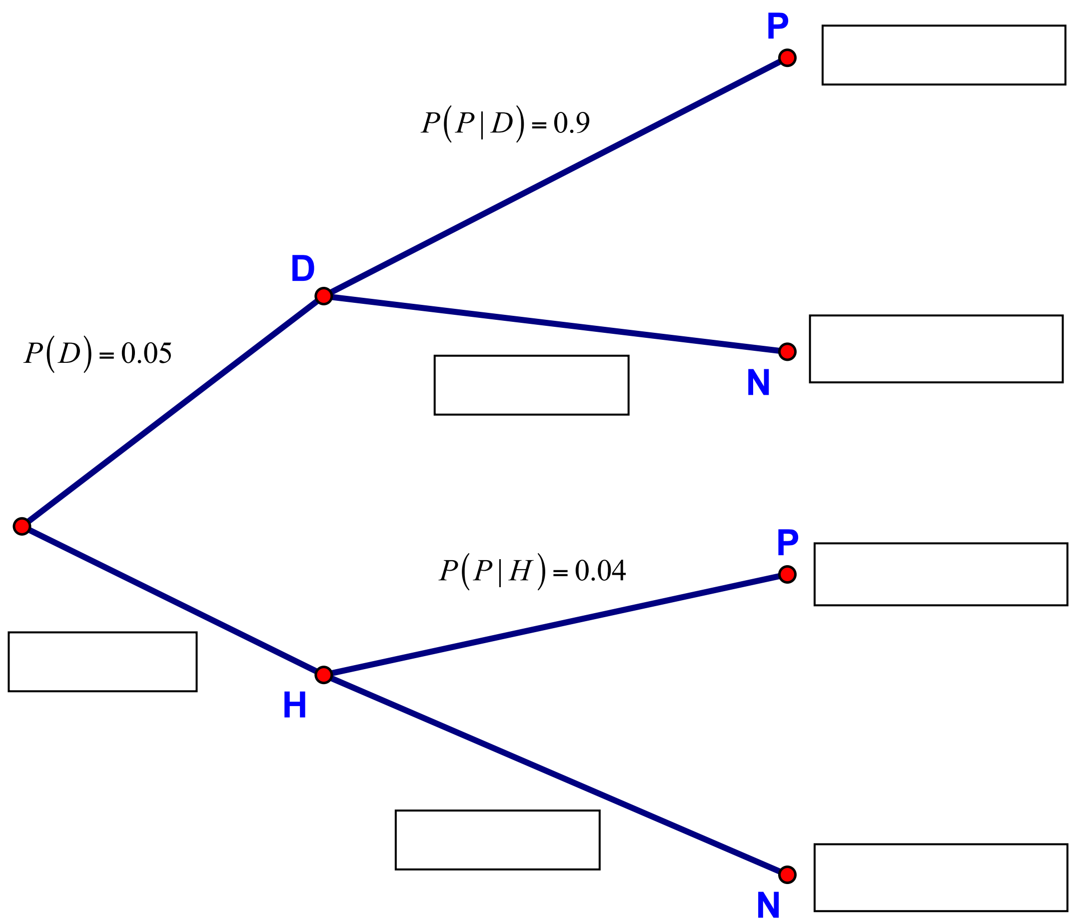
Suppose there is a disease that 5% of the population currently has. There is a test for this disease. 90% of those who have the disease test positive, and 4% of those who do not have the disease also test positive for the disease. A random person walks in for a wellness exam and is given the test for the disease. We use the following letters to represent events:

D = the person actually has the disease
H = the person does not have the disease (healthy)
P = the person tests positive for the disease
N = the person tests negative for the disease.

We can represent some of the probabilities of related events with the following probability tree. The given information has already been entered into the tree.

A. Complete the tree by computing and labeling the probability of each edge and each path through the tree. Fill in the values in each of the boxes below.

B. What is the probability that a random person will have the disease and test positive? 4.5%



ANSWERS:

1	A
2	B
3	D
4	C
5	B
6	B
7	A
8	C
9	D
10	D
11	C
12	C
13	C
14	C
15	A
16	D
17	C
18	D
19	E
20	A
21	D
22	D
23	B
24	B
25	A

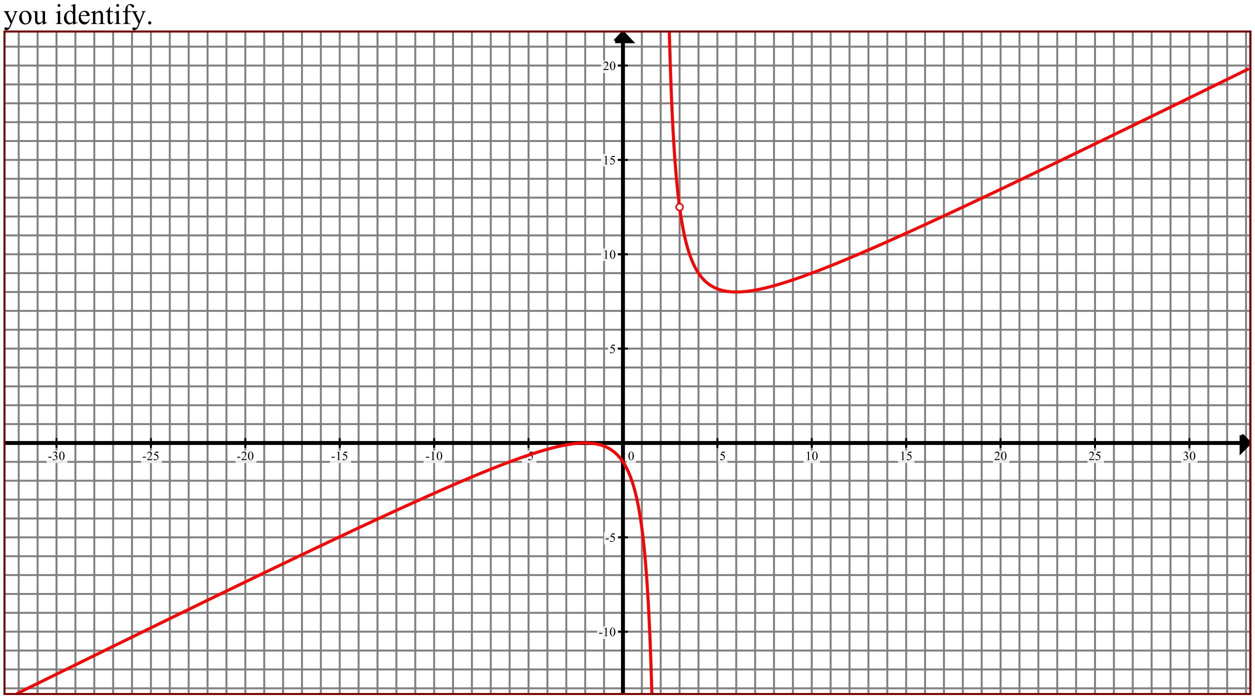
ACTM State Algebra II Competition 2015
Tie Breaker 1

Name **Solution**

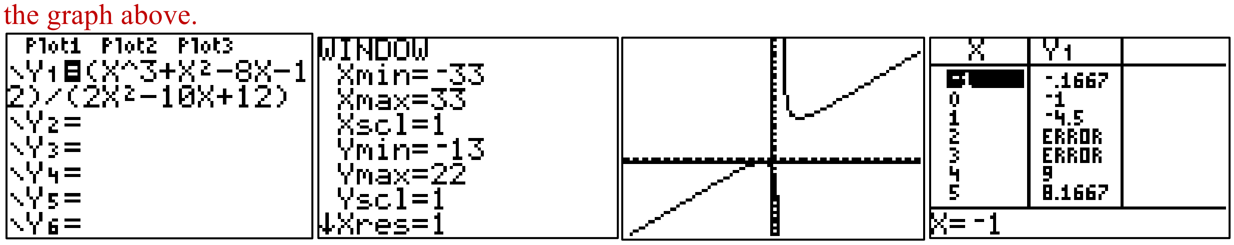
Clearly and carefully graph the function

$$f\left(x\right)=\frac{x^3+x^2-8x-12}{2x^2-10x+12}$$

on the grid below and then comment on important features of the graph including such things as domain and range, and any intercepts, extrema, holes, and asymptotes. Justify your conclusions for each feature you identify.



We can enter the original formula in the calculator and get a graph and a table to help plot the points on the graph above.



Notice error messages at x = 2 and x = 3 so (x-2) and (x-3) should be factors of the denominator. We can see the vertical asymptote at x = 2, but there is not a vertical asymptote at x = 3 so this must be a hole in the graph and (x-3) is a factor of the numerator as well. We can then factor and reduce the original function as follows:

$$f\left(x\right)=\frac{x^3+x^2-8x-12}{2x^2-10x+12}=\frac{\left(x-3\right)\left(x^2+4x+4\right)}{2\left(x^2-5x+6\right)}=\frac{\left(x-3\right)\left(x+2\right)^2}{2\left(x-3\right)\left(x-2\right)}=\frac{\left(x+2\right)^2}{2\left(x-2\right)},x\neq 3$$

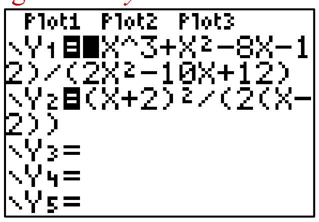
Note that we can factor out the (x-3) in the numerator using synthetic division:

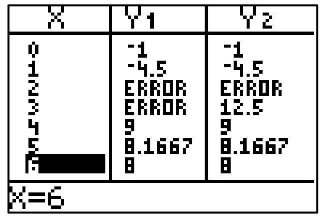
3	1	1	-8	-12
		3	12	12
	1	4	4	0

Also note that $\frac{(3+2)^2}{2(3-2)} = \frac{25}{2} = 12.5$.

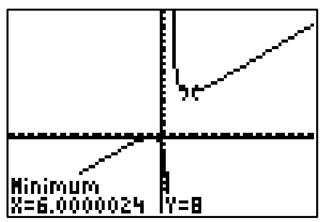
- So we see the domain is all real numbers except 2 and 3.
- There is a vertical asymptote at $x = 2$ and there is hole in the graph at (3, 12.5).

We can check the factoring and the y-value of the hole with the calculator.





- Note that the only x-intercept is (-2, 0) and -2 is a multiplicity 2 root so this point is also a local maximum.
- There is a local minimum at (6, 8) which we can find with the CALC Minimum feature of the calculator.

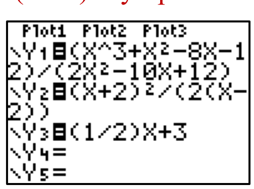


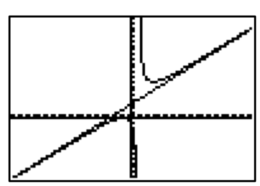
- So we see that the range is (-∞, 0] ∪ [8, ∞).
- Notice that the function is concave down to the left of the vertical asymptote at $x = 2$ and it is concave up to the right of $x = 2$.
- The function is increasing on (-∞, -2) and on (6, ∞) and is decreasing on x-values of (-2, 2), (2, 3), and (3, 6).

Since the degree of the numerator is one larger than the degree of the denominator there is an oblique asymptote which we can find by long division.

$$\begin{array}{r} \frac{1}{2}x + 3 \\ 2x-4 \overline{\big)\, x^2 + 4x + 4} \\ \underline{\boxed{-}x^2 \boxed{+} -2x} \\ 6x + 4 \\ \underline{\boxed{-}6x \boxed{+} -12} \\ 16 \end{array}$$

- So $y = \frac{1}{2}x + 3$ is an oblique (slant) asymptote.





ACTM State Algebra II Competition 2015
Tie Breaker 2

Name **Solution**

The first two terms of a sequence are 4 and 7 in that order. When we find the differences between successive terms this produces a sequence of first differences. When we find the difference between successive terms of the sequence of first differences we obtain a sequence of second differences. The sequence of second differences is a constant sequence with value 4.

A. Write out the first 8 output values of the original sequence in order.

We are given that the first two terms are 4 and 7. This produces a first difference of 7 – 4 = 3. The second difference is always 4 so the sequence of first differences is then:

3, 3+4=7, 7+4=11, 11+4 = 15, 15+4=19, 19+4=23, 23+4=27, 27+4=31,…
= 3, 7, 11, 15, 19, 23, 27, 31, ...

Therefore the original sequence is

4, 4+3=7, 7+7=14, 14+11=25, 25+15=40, 40+19=59, 59+23=82, 82+27=109, …

The original sequence is

$\boxed{\textbf{4, 7, 14, 25, 40, 59, 82, 109}}$, …

(Sequence of first differences: 3, 7, 11, 15, 19, 23, 27, …
Sequence of second differences: 4, 4, 4, 4, 4, 4, …)

B. Write out an explicit non-recursive formula for the k^{th} term of the original sequence.

A sequence with a common second difference is a quadratic sequence. So this sequence has a formula of $d_k = ak^2 + bk + c$. (1, 4), (2, 7), and (3, 14) are points of this sequence so

$$a(1)^2 + b(1) + c = 4 \quad \Rightarrow \quad a + b + c = 4$$
$$a(2)^2 + b(2) + c = 7 \quad \Rightarrow \quad 4a + 2b + c = 7$$
$$a(3)^2 + b(3) + c = 14 \quad \Rightarrow \quad 9a + 3b + c = 14$$

From here we can proceed by several methods.

Method 1: $d_1 = a + b + c = 4$ $\qquad d_2 = 4a + 2b + c = 7$ $\qquad d_3 = 9a + 3b + c = 14$

1st differences: $\qquad 3a + b = 3 \qquad 5a + b = 7$

2nd difference $\qquad 2a = 4$

So, a = 2 (students may know a is ½ of the 2nd difference), then $3(2) + b = 3 \Rightarrow b = -3$ and then $2 - 3 + c = 4 \Rightarrow c = 5$.

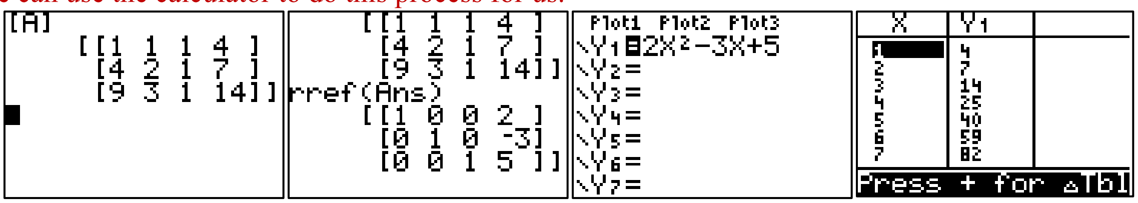
Method 2: We can solve this system of linear equations to find a, b, and c. One way to do this is using Gauss-Jordan Matrix reduction.

$$\begin{bmatrix}1&1&1&4\\4&2&1&7\\9&3&1&14\end{bmatrix}\xrightarrow{R_2-4R_1}\begin{bmatrix}1&1&1&4\\0&-2&-3&-9\\9&3&1&14\end{bmatrix}\xrightarrow{R_3-9R_1}\begin{bmatrix}1&1&1&4\\0&-2&-3&-9\\0&-6&-8&-22\end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2}\begin{bmatrix}1&1&1&4\\0&1&\frac{3}{2}&\frac{9}{2}\\0&-6&-8&-22\end{bmatrix}\xrightarrow{R_3+6R_2}\begin{bmatrix}1&1&1&4\\0&1&\frac{3}{2}&\frac{9}{2}\\0&0&1&5\end{bmatrix}\xrightarrow{R_2-\frac{3}{2}R_3}\begin{bmatrix}1&1&1&4\\0&1&0&-3\\0&0&1&5\end{bmatrix}$$

$$\xrightarrow{R_1-R_3}\begin{bmatrix}1&1&0&-1\\0&1&0&-3\\0&0&1&5\end{bmatrix}\xrightarrow{R_1-R_2}\begin{bmatrix}1&0&0&2\\0&1&0&-3\\0&0&1&5\end{bmatrix}$$

We can use the calculator to do this process for us.



Method 3: Quadratic regression in calculator using the three points.

So the equation for the sequence is $\boxed{d_k = 2k^2 - 3k + 5}$.

ACTM State Algebra II Competition 2015
Tie Breaker 3

Name **Solution**

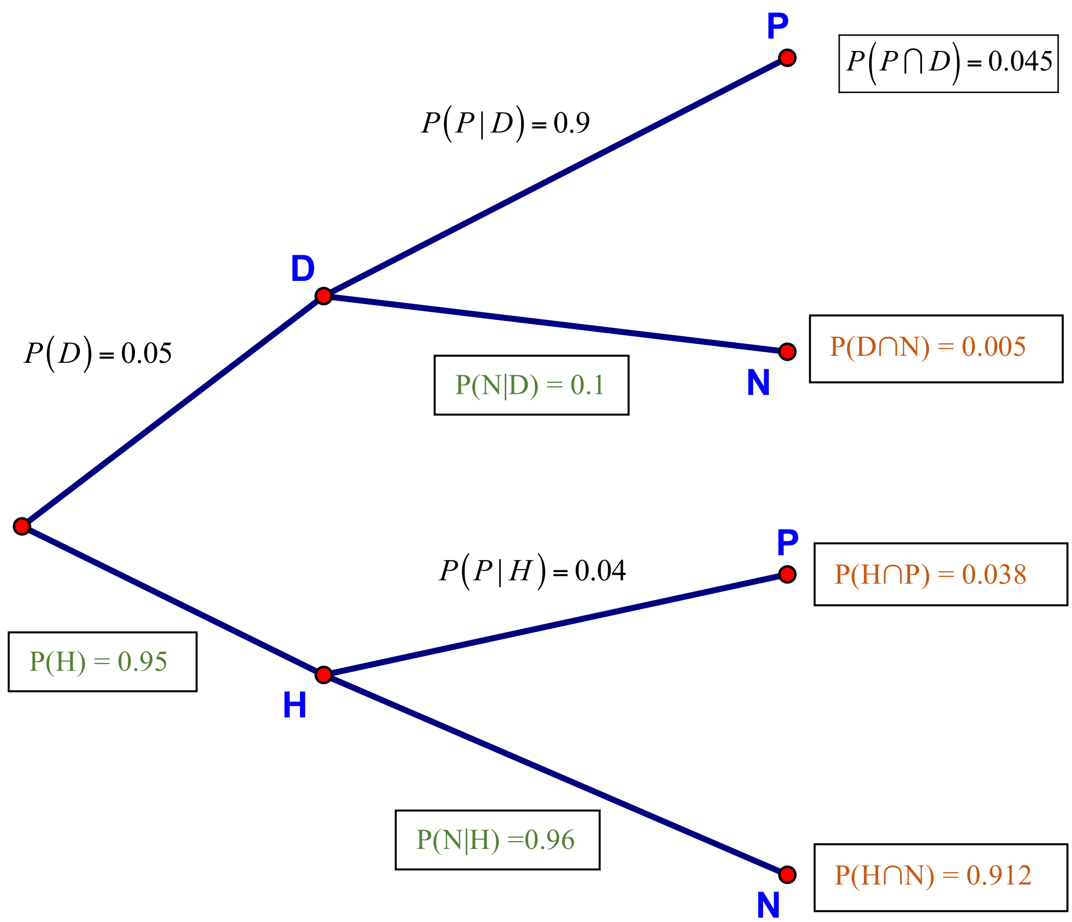
Suppose there is a disease that 5% of the population currently has. There is a test for this disease. 90% of those who have the disease test positive, and 4.526% of those who do not have the disease also test positive for the disease. A random person walks in for a wellness exam and is given the test for the disease. We use the following letters to represent events:

D = the person actually has the disease
H = the person does not have the disease (healthy)
P = the person tests positive for the disease
N = the person tests negative for the disease.

We can represent some of the probabilities of related events with the following probability tree. The given information has already been entered into the tree.

A. Complete the tree by computing and labeling the probability of each edge and each path through the tree. Fill in the values in each of the boxes below.

B. What is the probability that a random person will have the disease and test positive? 4.5%



- First note that the sum of all probabilities leaving any vertex must be 1. So we can complete the probabilities of all the edges in the first tree as follows.
  - $P(H) = 1 - P(D) = 1 - 0.05 = 0.95$
  - $P(N|D) = 1 - P(P|D) = 1 - 0.9 = 0.1$
  - $P(N|H) = 1 - P(P|H) = 1 - 0.04 = 0.96$
- Next probabilities of a path through the tree (intersection of events) are found by multiplying the probabilities along the path. This allows us to complete the tree on the left.
  - $P(D \cap P) = P(D)P(P|D) = (0.05)(0.9) = 0.045$
  - $P(D \cap N) = P(D)P(N|D) = (0.05)(0.1) = 0.005$
  - $P(H \cap P) = P(H)P(P|H) = (0.95)(0.04) = 0.038$
  - $P(H \cap N) = P(H)P(N|H) = (0.95)(0.96) = 0.912$