ACTM State Precalculus Competition
Spring 2014

Begin by removing the three tie breaker sheets at the end of the exam and writing your name on all three pages. Work the 22 multiple choice questions first choosing the single best response from the choices available. Indicate your answer here and on your answer sheet. Make sure you attempt the tie-breaker questions at the end of the test if you have time. Turn in your answer sheet and the tie breaker pages when you are finished. You may keep the pages with the multiple choice questions.

1. You purchase a new automobile for $15,200. (That is the value of the automobile when it is driven off the lot is $15,200.) It depreciates 17.2% per year. What is the value of the car three years after it is purchased?

   A. $9070

   B. $11,300

   C. $7250

   D. $8520

   E. $8630

2. tan(arccos(x/4)) =

   A. $\frac{\sqrt{16-x^2}}{x}$

   B. $\frac{x}{\sqrt{16+x^2}}$

   C. $\frac{x}{4}$

   D. $\frac{\sqrt{16+x^2}}{x}$

   E. Each of the other answers is incorrect.

3. Sales of a particular beverage are modeled by the following function where t is the number of days since the beginning of 2010.

$$S(t) = 3752e^{0.05t} + 650\sin\left(\tfrac{2\pi}{365}(t-60)\right)$$

   Which of the following best describes these beverage sales?

   A. Sales are basically constant.

   B. Sales are increasing at around 5% per year.

   C. Sales are periodic, going up and down throughout the year.

   D. There is a year to year increasing pattern with a seasonal up and down cycle within it.

   E. Sales are decreasing at around 5% per year.

4. Consider the graph of $f(x) = -3\sin(\pi(x-2)) + 4$. Which of the following is NOT correct?

   A. This function is periodic with period 2.
   B. The range of this function is [-7,1].
   C. The y-intercept of this function is (0, 4).
   D. The minimum output of this function is 1.
   E. The amplitude of this function is 3.

5. Which of the following is NOT an identity for positive values of x, y and b?

   A. $\ln(xy) = \ln(x) + \ln(y)$
   B. $\ln(x^n) = n\ln(x)$
   C. $\ln\left(\frac{x}{y}\right) = \dfrac{\ln(x)}{\ln(y)}$
   D. $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$
   E. $\log_b(1) = 0$

6. Determine the area of the triangle with vertices at the origin and two foci of the ellipse below.

$$16x^2 - 64x + 25y^2 - 50y = 311$$

   A. 3 square units
   B. 16 square units
   C. 2 square units
   D. 4 square units
   E. 5 square units

7. Consider the functions $f(x) = \frac{1}{1+x}$ and $g(x) = \frac{1}{x} - 1$. Let

$$H(x) = (f \circ g \circ f)(x).$$

Find $(H \circ H)(1)$.

A. $\frac{1}{2}$

B. $\frac{3}{5}$

C. 1

D. $\frac{2}{3}$

E. -1

8. Find

$$\frac{(1+i)^{2014}}{2^{1007}}$$

A. $-1+i$

B. -1

C. $\frac{1-i}{2}$

D. 1

E. $-i$

9. A clock has a three inch hour-hand and a four inch minute-hand. How much further apart are the hands at 9:00 am than at 10:00 am? Round this answer to two decimal places.

A. 1.39 inches

B. 4.93 inches

C. 0.78 inches

D. 2.05 inches

E. 3.61 inches

10. A manufacturer of office supplies makes three types of file cabinet: two-drawer, four-drawer, and horizontal. The manufacturing process is divided into three phases: assembly, painting, and finishing. A two-drawer model takes 3 hours to assemble, 1 hour to paint, and 1 hour to finish. A four-drawer model takes 5 hours to assemble, 90 minutes to paint, and 2 hours to finish. The horizontal cabinet takes 4 hours to assemble, 1 hour to paint, and 3 hours to finish. The manufacturer employs enough workers for 500 hours of assembly time, 150 hours of painting, and 230 hours of finishing per week. Determine how many of each type of file cabinet the manufacturer should make in order to use all of the worker-hours available. What is the total number of cabinets made in this optimal situation?

    A. 100

    B. 130

    C. 150

    D. 200

    E. Each of the other answers is incorrect.

11. Let $M = \begin{bmatrix} 2x & 3 \\ -x & x \end{bmatrix}$. M^{-1} exists when:

    A. $x \in (-\infty, \infty)$

    B. $x \in (-\infty, 0) \cup (0, \infty)$

    C. $x \in (-\infty, -1.5) \cup (-1.5, 0) \cup (0, \infty)$

    D. $x \in (-\infty, 0) \cup (0, 1.5) \cup (1.5, \infty)$

    E. Each of the other answers is incorrect.

12. If for some real numbers a and b

$$\frac{7x-1}{x^2-x-6} = \frac{a}{x+2} + \frac{b}{x-3}$$

then a =

    A. -2

    B. 2

    C. 3

    D. 4

    E. Each of the other answers is incorrect.

13. Use a power reducing identity to rewrite the following expression without the power of two:
$3\sin^2(4x) =$

A. $3 - 3\cos^2(4x)$

B. $3 - 3\cos(8x)$

C. $\frac{3}{2}(1 - \cos(8x))$

D. $\frac{3}{2}(1 + \cos(8x))$

E. $\frac{3}{2}(1 - \sin(8x))$

14. Simplify the following as far as possible using reduced radicals:

$$3\sqrt{20} - 4\sqrt{45} - \sqrt[4]{25} =$$

A. $-6\sqrt{5} - \sqrt[4]{25} =$

B. $15\sqrt{2} - 20\sqrt{3} - \sqrt{5}$

C. $5\sqrt{5}$

D. $-7\sqrt{5}$

E. Each of the other answers is incorrect.

15. Completely simplify the following using reduced radicals and rationalized denominators:

$$\frac{3x^2y}{\sqrt[3]{32x^4y^6z^{14}}} =$$

A. $\frac{3x^2y\sqrt[3]{2x^2z}}{4z^5}$

B. $\frac{3x^2y\sqrt[3]{4xz^2}}{2z^5}$

C. $\frac{3\sqrt[3]{2x^2z}}{4yz^5}$

D. $\frac{3x^2y\sqrt[3]{2xz}}{4z^5}$

E. Each of the other answers is incorrect.

16. Consider the graph of the function $f(x) = x^{\frac{p}{q}}$ where p is an integer, q a natural number and the fraction p/q is reduced. If we know that the graph is in quadrant I only and is increasing concave up what can we say about p and q?

    A. p is positive, $p < q$, p is even, and q is odd.

    B. p is positive, $p > q$, p is even, and q is odd.

    C. p is positive, $p > q$, p is odd, and q is even.

    D. p is positive, $p < q$, p is odd, and q is even.

    E. Each of the other answers is incorrect.

17. What is the coefficient of the third degree term of $(2x-3)^5$?

    A. 10

    B. 72

    C. 80

    D. 720

    E. Each of the other answers is incorrect.

18. What are the asymptotes of the graph of $f(x) = \dfrac{2x^3 - 54}{x^2 + 3x + 2}$?

    A. $x = -2$ and $x = -1$

    B. $x = -2$, $x = -1$, and $y = 2$

    C. $x = -2$, $x = -1$, and $y = 2x$

    D. $x = -2$, $x = -1$, and $y = 2x-6$

    E. Each of the other answers is incorrect.

19. A person in a boat measures the angle of inclination from 6.00 feet above the surface of the water to the top of a cliff to be 25.0^o. The boat then moves 650 feet closer to the cliff and the angle of inclination now measures 42.0^o. To the nearest foot, how high is the top of the cliff above the water level?

    A. 581 feet

    B. 635 feet

    C. 587 feet

    D. 629 feet

    E. Each of the other answers is incorrect.

20. A soft drink company is making a cylindrical aluminum can of radius *r*. The volume of the can is 355 cm^3. In order to accommodate the pull tab the top of the can is three times as thick as the sides and bottom of the can. To the nearest tenth of a centimeter what radius should be used to minimize the amount of aluminum needed for the can?

A. 3.0 cm

B. 4.8 cm

C. 5.1 cm

D. 6.2 cm

E. Each of the other answers is incorrect.

21. To the nearest degree what is the angle between the vectors $\langle 3,5 \rangle$ and $\langle -5,7 \rangle$?

A. 5^o

B. 72^o

C. 64^o

D. 67^o

E. Each of the other answers is incorrect.

22. The two complex numbers $\sqrt{2} + \sqrt{2}i$ and $\sqrt{3} - i$ lie on the circle centered at the origin with radius two in the complex plane. The two numbers divide the circle into two arcs. Find the length of the smaller arc.

A. 4π units

B. $\frac{5\pi}{6}$ units

C. $\frac{5\pi}{12}$ units

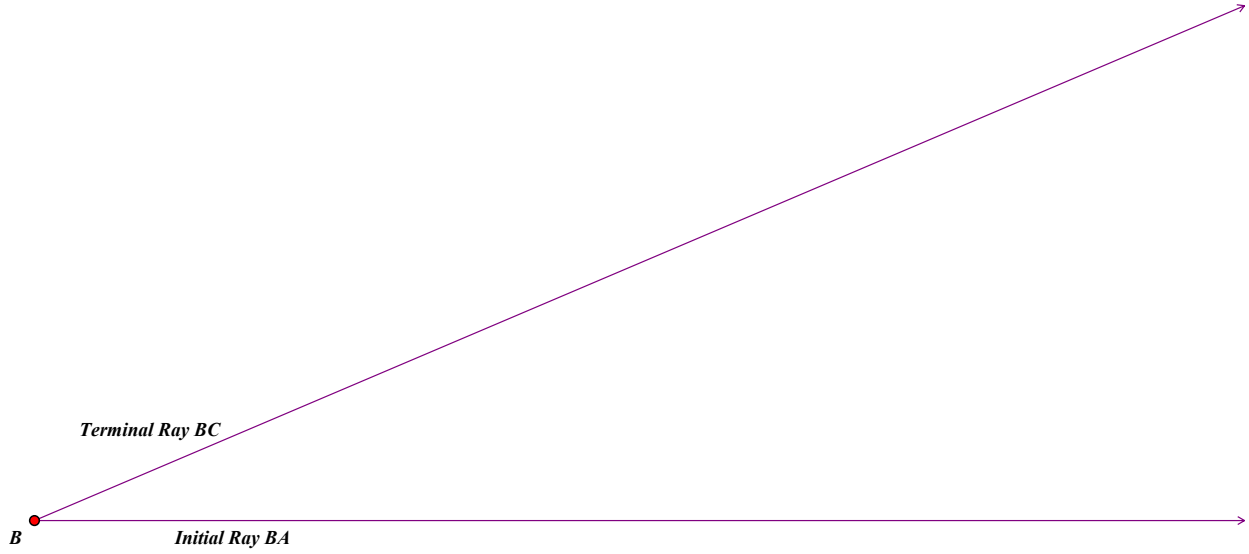
D. $\frac{2\pi}{3}$ units

E. $\frac{7\pi}{6}$ units

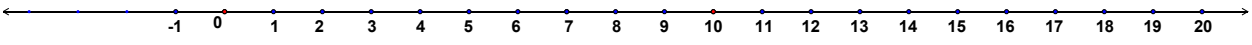
ACTM State Precalculus Competition 2014
Tie Breaker 1

Name _____

Explain how you can use the number line/ruler on bottom of this page to calculate the tangent of the following angle to two decimal places without using a calculator. Use your method to compute this tangent to two decimal places.



You may tear off the ruler for Tie Breaker 1:



ACTM State Precalculus Competition 2014
Tie Breaker 2

Name _____

Let $S(x)$ be the number of sunlight hours on a cloudless June 21, the summer solstice, as a function of the latitude x measured in degrees. Recall that the equator has latitude 0^o and the North Pole has latitude 90^o. 66.5^o is the latitude of the Arctic Circle above which the Sun does not set on June 21. In the Northern Hemisphere *S(x)* is given by

$$S(x) = \begin{cases} a + b\arcsin\left(\dfrac{\tan(x)}{\tan(66.5^o)}\right) & \text{for } x \in [0^o, 66.5^o] \\ 24 & \text{for } x \in [66.5^o, 90^o] \end{cases}$$

where we use degrees for the output of the arcsine function.

A. Note that the equator always has equal amounts of daylight and night year round. Use this to compute a.

B. The function is continuous. Use this fact to compute b.

C. Compute *S(x)* for Tucson, AZ ($x = 32^o\ 13'$),

ACTM State Precalculus Competition 2014
Tie Breaker 3

Name _____

Consider the following simultaneous system of equations.

$$y = x^3$$

$$y = 3^x$$

A. How many solutions does this system have?

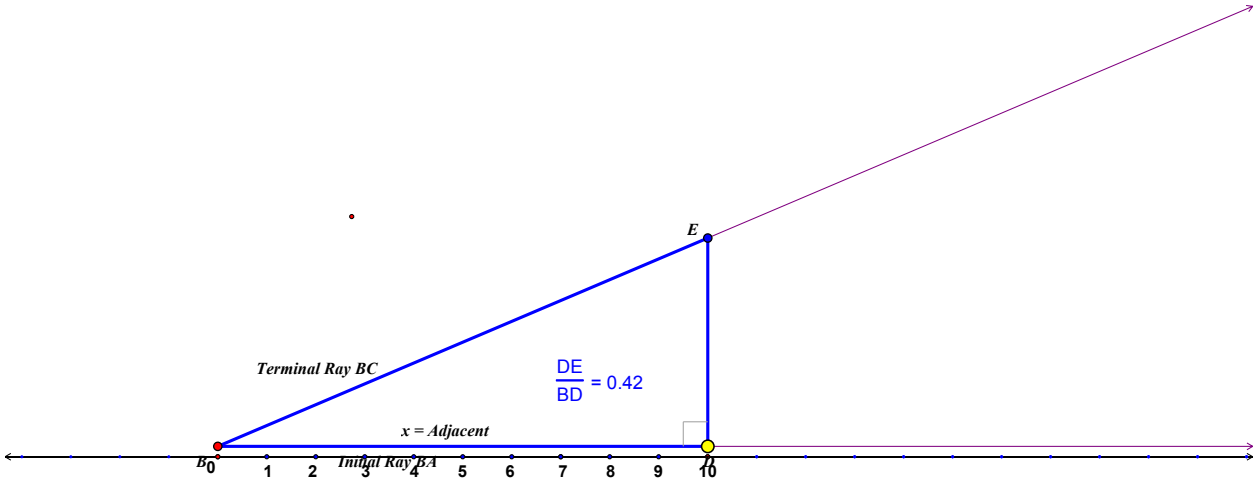
B. Approximate all of the solutions to the system of equations to four decimal places.

ACTM State Precalculus Competition 2014
Tie Breaker 1

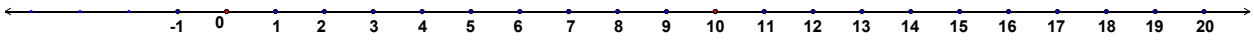
Name **Solution Key**

Explain how you can use the number line/ruler on top of the next page to calculate the tangent of the following angle to two decimal places without using a calculator. Use your method to compute this tangent to two decimal places.

In order to compute the tangent of this angle we need to construct a right triangle with this angle as one of the interior angles. Then by definition the tangent of the angle is the measure of the length of the opposite leg divided by the length of the adjacent leg. Since we are instructed to perform this division without a calculator, and we have free choice for the length of the adjacent leg we will choose 10 units for this length since it is easy to divide by. In this case we see that the length of the opposite leg is then approximately 4.2 units long so the tangent of this angle is then approximately 4.2/10 = **0.42**.



You may tear off the ruler for Tie Breaker 1:



ACTM State Precalculus Competition 2014
Tie Breaker 2

Name _____

Let $S(x)$ be the number of sunlight hours on a cloudless June 21, the summer solstice, as a function of the latitude x measured in degrees. Recall that the equator has latitude 0^o and the North Pole has latitude 90^o. 66.5^o is the latitude of the Arctic Circle above which the Sun does not set on June 21. In the Northern Hemisphere $S(x)$ is given by

$$S(x) = \begin{cases} a + b\arcsin\left(\dfrac{\tan(x)}{\tan(66.5^o)}\right) & \text{for } x \in [0^o, 66.5^o] \\ 24 & \text{for } x \in [66.5^o, 90^o] \end{cases}$$

where we use degrees for the output of the arcsine function.

A. Note that the equator always has equal amounts of daylight and night year round. Use this to compute a.
Since there are 24 hours per day and the equator has an equal number of hours of light and dark each day of the year $S(0) = 12$.

$$12 = S(0) = a + b\arcsin\left(\frac{\tan(0)}{\tan(66.5^o)}\right)$$
$$12 = a + b\arcsin\left(\frac{0}{\tan(66.5^o)}\right)$$
$$12 = a + b\arcsin(0)$$
$$12 = a + b(0)$$
$$12 = a$$

B. The function is continuous. Use this fact to compute b.
Since this function is continuous $S(66.5^o) = 24$ using either piece of the formula. So:

$$S(66.5^o) = 24 = 12 + b\arcsin\left(\frac{\tan(66.5^o)}{\tan(66.5^o)}\right)$$
$$12 = b\arcsin(1)$$
$$12 = b90^o$$
$$b = \frac{12 hours}{90^o}$$
$$b = \frac{2\ hours}{15^o}$$

So the formula for S is

$$S(x) = \begin{cases} 12 + \left(\frac{2}{15}\right)\arcsin\left(\dfrac{\tan(x)}{\tan\left(66.5^{o}\right)}\right) & \text{for } x \in \left[0^{o}, 66.5^{o}\right] \\ 24 & \text{for } x \in \left[66.5^{o}, 90^{o}\right] \end{cases}$$

C. Compute *S(x)* for Tucson, AZ ($x = 32^{o}\ 13'$).

$32^{o}\ 13' = (32 + 13/60)^{o}$

$$S\left(\left(32 + \tfrac{13}{60}\right)^{o}\right) = 12 + \left(\tfrac{2}{15}\right)\arcsin\left(\frac{\tan\left(\left(32 + \frac{13}{60}\right)^{o}\right)}{\tan\left(66.5^{o}\right)}\right)$$

$$\approx 14.12 \textit{hours} \approx 14 \textit{ hours } 7 \textit{ minutes}$$

ACTM State Precalculus Competition 2014
Tie Breaker 3

Name _____

Consider the following simultaneous system of equations.

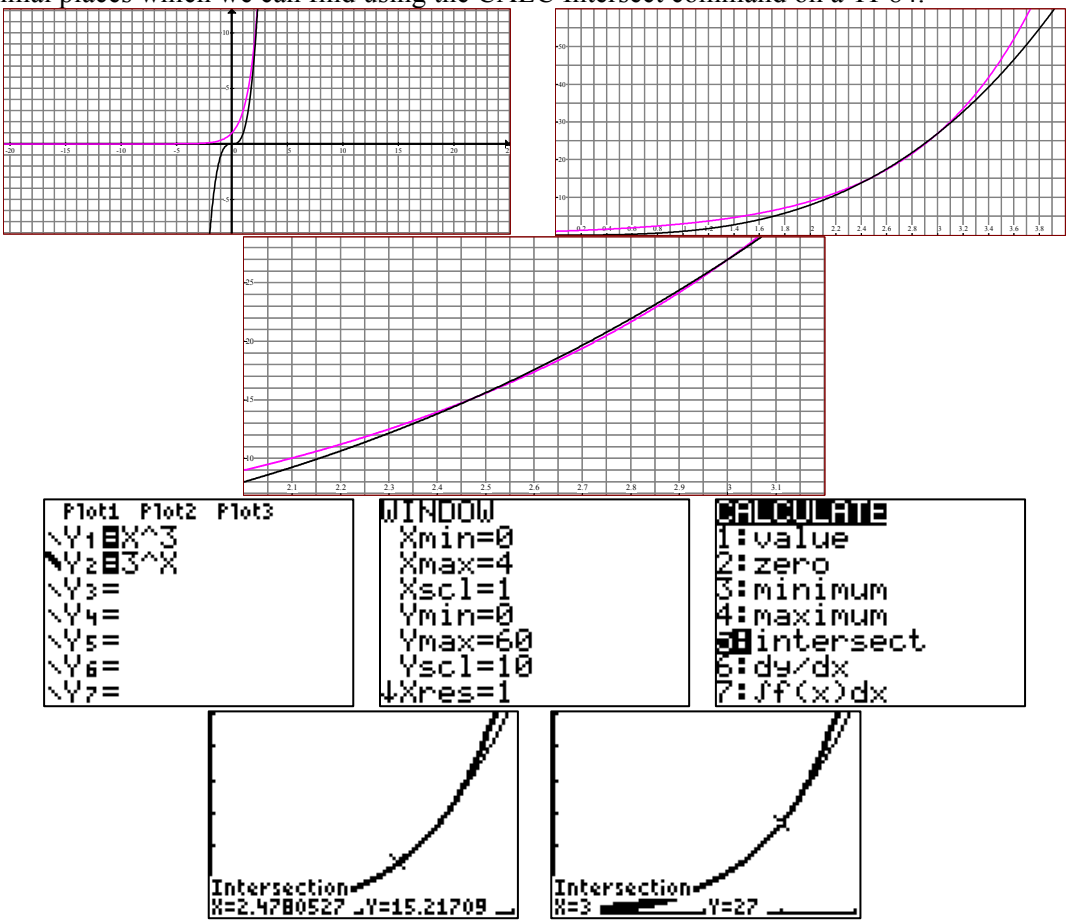
$$y = x^3$$

$$y = 3^x$$

A. How many solutions does this system have?

This system has two solutions. Note that the graph of the exponential function $y = 3^x$ is above the graph of the power function $y = x^3$ for $x \leq 0$. They first cross each other at a point with x between 0 and 3 (around 2.5) after which the power function is on the top for a while. However, we should know that eventually there will be a point where the exponential will be larger than the power function. Therefore they cross one more time at $(x, y) = (3, 27)$ after which the exponential function stays on top.

B. Approximate all of the solutions to the system of equations to four decimal places.
$(x, y) = (3, 27)$ is an exact solution. $(x, y) = (2.4781, 15.2171)$ is an approximate solution to four decimal places which we can find using the CALC Intersect command on a TI-84.



ANSWERS:

1. E
2. A
3. D
4. B
5. C
6. A
7. D
8. E
9. A
10. B
11. C
12. C
13. C
14. D
15. C
16. C
17. D
18. D
19. B
20. A
21. D
22. B