State Precalculus/Trigonometry Contest 2008

Select the best answer for each of the following questions and mark it on the answer sheet provided. Be sure to read all the answer choices before making your selection. When you are finished with the multiple choice, attempt the tiebreaker questions.

1. The exact value of $\cot \frac{\pi}{3}$ is

a) $\frac{1}{2}$ b) $\frac{\sqrt{3}}{3}$ c) $\frac{2\sqrt{3}}{3}$ d) 2 e) $\sqrt{3}$

2. If θ is an angle in standard position whose terminal side passes through the point $(-5, 3)$, then the value of $\sin\theta$ is

a) $\frac{3}{5}$ b) $-\frac{3}{5}$ c) $\frac{4}{5}$ d) $-\frac{4}{5}$ e) $\frac{3\sqrt{34}}{34}$

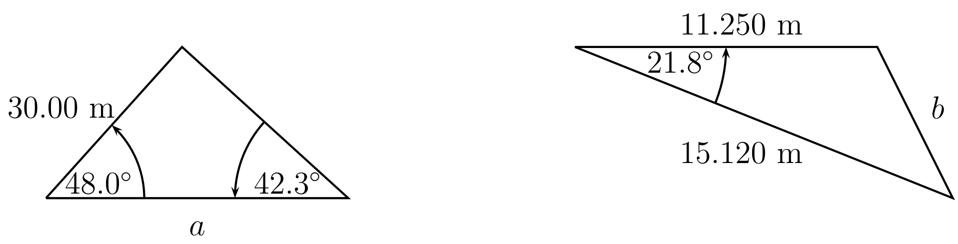

Figure 1: Triangles for Problems 3-4

3. The length of side a in Figure 1 is closest to

a) 33.1 m b) 33.3 m c) 38.0 m d) 42.5 m e) 44.6 m

4. The length of side b in Figure 1 is closest to

a) 7.90 m b) 7.36 m c) 6.52 m d) 6.46 m e) 6.27 m

5. The smallest angle of a triangle with sides of lengths 4.00 ft, 7.00 ft, and 10.00 ft, has a degree measure closest to

a) 9.6° b) 14.7° c) 17.0° d) 18.2° e) 21.2°

6. If the legs of a right triangle are of lengths 6.000 m and 9.000 m, then the measure of the larger acute angle of the triangle is closest to

a) 56.31° b) 42.00° c) 37.65° d) 33.69° e) 28.20°

7. If a right triangle has a hypotenuse of length 12.000 m and a longer leg of length 9.000 m, then the length of the shorter leg to the nearest hundredth of a meter is

a) 6.00 m b) 7.21 m c) 7.56 m d) 7.94 m e) 8.46 m

8. Pick the correct summarization of the accuracy of the three statements

I. $\sin^2 x = 1 + \cos^2 x$, II. $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ III. $\cos^2 x = 2\sin x \cos x$

a) I, II, and III are true for all x b) Only II is true for all x c) None of the statements are true for all x d) I and II are true for all x but III is not e) Only III is true for all x

9. The graph of $y = \sin x$ is nothing more than the graph of $y = \cos x$ shifted

a) to the right π units b) to the right $\frac{\pi}{2}$ units c) to the right $\frac{\pi}{4}$ units d) to the left $\frac{\pi}{4}$ units e) to the left $\frac{\pi}{2}$ units

10. Which of the following pairs of functions f and g do not have the same domain?

a) $f(x) = \cos x$ and $g(x) = \sin x$ b) $f(x) = \tan x$ and $g(x) = \cot(x + \frac{\pi}{2})$ c) $f(x) = \ln x$ and $g(x) = \frac{1}{\sqrt{x}}$ d) $f(x) = \sec x$ and $g(x) = \csc x$ e) $f(x) = e^x$ and $g(x) = \sin x$

11. The function $y = \sin x + \cos x$ can be written as $y = c\sin(x + \frac{\pi}{4})$ where c has the value

a) $\sqrt{2}$ b) 2 c) $\sqrt{3}$ d) 4 e) -1.5

12. If $\cos x = 0.6$, $\sin y = 0.5$, $\dfrac{3\pi}{2} \le x \le 2\pi$, and $0 \le y \le \dfrac{\pi}{2}$, then the value of $\cos(x + y)$ is

a) $\dfrac{\sqrt{3}}{10}$ b) 0.1 c) 1.1 d) $\dfrac{4 + 3\sqrt{3}}{10}$ e) $\dfrac{3 + 4\sqrt{3}}{10}$

13. For the circle with polar coordinates equation $r = 4\sin\theta$, the center in rectangular coordinates is

a) $(0, 0)$ b) $(4, 0)$ c) $(0, 4)$ d) $(2, 2)$ e) $(0, 2)$

14. The solution of the equation $\log_3 x = -2$ is

a) $x = -6$ b) $x = \frac{1}{6}$ c) $x = -\frac{3}{2}$ d) $x = -\frac{2}{3}$ e) $x = \frac{1}{9}$

15. Suppose you invest money at a 5% annual interest rate in an account that has continuously compounding interest. What's the least amount you could invest today so that you'll have $1,214.87 in the account in six years?

a) $893.45 b) $900.00 c) $906.55 d) $925.00 e) $950.00

16. The complex number $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^{20}$ in $a + bi$ form is

a) -1 b) 0 c) 1 d) $\dfrac{20}{\sqrt{2}} + \dfrac{20i}{\sqrt{2}}$ e) $10 + 10i$

17. The line that is perpendicular to $4x + 8y = 0$ and passes through the point (1,3) also passes through the point

a) $(-4, -7)$ b) (0,0) c) $(1, -\frac{1}{2})$ d) (2,1) e) (4,5)

18. If the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, where a, b, c, d, and e, are real constants, has the solutions $2 + 3i$ and $16 - i$, then how many real solutions does the equation have?

a) none b) exactly one c) exactly two d) exactly three e) infinitely many

19. If $f(x) = \dfrac{2x-3}{x+1}$, then the inverse function $f^{-1}(x)$ of $f(x)$ is given by

a) $f^{-1}(x) = \dfrac{\frac{1}{2}x+3}{x-1}$ b) $f^{-1}(x) = \dfrac{x+1}{2x-1}$ c) $f^{-1}(x) = (2x-3)(x+1)$ d) $f^{-1}(x) = \dfrac{x+3}{2-x}$ e) $f^{-1}(x) = (\dfrac{1}{2}x - 3)(x-1)$

20. The following system of equations has how many solutions

$$\begin{aligned} 4x + 2y + 3z &= 4 \\ 8x + 4y + 6z &= 16 \\ 2x + 2y + 3z &= 5 \end{aligned}$$

a) none b) exactly one c) exactly two d) exactly three e) infinitely many

21. Suppose the set A has only 50 elements in it. The number of subsets of A with either three or four elements is then

a) 9 b) 652 c) $5,642$ d) $249,900$ e) $5,644,800$

22. The geometric series $\dfrac{2}{3} - \dfrac{4}{9} + \dfrac{8}{27} - \dfrac{16}{81} + \cdots$ has the sum

a) $\frac{32}{243}$ b) 0 c) 1 d) $\frac{2}{5}$ e) $\frac{3}{5}$

23. If the parabola $y = ax^2 + bx + c$ passes through the three points $(-2, -5)$, $(0, 5)$, and $(1, 4)$, then the value of a is

a) -2 b) -1 c) 2 d) 3 e) 4

24. An equivalent expression for $(e^x)^2 + \ln(x^2+3) + \ln e^x$ is

a) $e^{x^2} + \ln(x^2+3) + x$ b) $e^{2x} + \ln(x^2+3) + x$ c) $e^{2x} + 2\ln x + \ln 3 + x$ d) $e^{2x} + 2\ln x + \ln 3 + e\ln x$ e) $e^{x^2} + 2\ln x + \ln 3 + x$

25. If $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then A^2 is

a) $\begin{pmatrix} 1 & 4 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 6 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

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ACTM State Precalculus/Trigonometry Contest, 2008

Tiebreakers

1. Use the Principle of Mathematical Induction to prove that $1+2+2^2+\cdots+2^{n-1}=2^n-1$ is true for all positive integers n.

2. Find the partial fraction decomposition of $\dfrac{14}{x^2+3x-10}$.

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3. Compute to the nearest hundredth of a square meter the area of a triangle with sides of lengths 9.000 m, 7.000 m, and 4.000 m.

Key for the State Precalculus/Trigonometry Contest 2008

1. B
2. E
3. E
4. E
5. D
6. A
7. D
8. B
9. B
10. D
11. A
12. D
13. E
14. E
15. B
16. A
17. A
18. A
19. D
20. A
21. D
22. D
23. A
24. B
25. B

Tiebreaker 1.

If $n = 1$, then $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^0 = 1$ and $2^n - 1 = 2^1 - 1 = 1$. Hence, $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$ when $n = 1$.

Suppose now that $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$ for some positive integer n. Then

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^{(n+1)-1} &= 1 + 2 + 2^2 + \cdots + 2^n \\ &= 1 + 2 + 2^2 + \cdots + 2^{n-1} + 2^n \\ &= 2^n - 1 + 2^n \\ &= 2 \cdot 2^n - 1 \\ &= 2^{n+1} - 1 \end{aligned}$$

Thus, by the Principle of Mathematical Induction, $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$ is true for all positive integers n.

Tiebreaker 2.

Since $x^2 + 3x - 10 = (x - 2)(x + 5)$, let

$$\frac{14}{x^2 + 3x - 10} = \frac{a}{x - 2} + \frac{b}{x + 5}.$$

Then multiplying each side by $(x - 2)(x + 5)$ we get

$$14 = a(x + 5) + b(x - 2).$$

Letting $x = 2$, we obtain $14 = 7a + 0$ or $a = 2$. By letting $x = -5$, we obtain $14 = 0 - 7b$ or $b = -2$. Hence,

$$\frac{14}{x^2 + 3x - 10} = \frac{2}{x - 2} - \frac{2}{x + 5}.$$

Tiebreaker 3.

If θ is the angle formed by the sides of lengths 4 and 7, then

$$9^2 = 4^2 + 7^2 - 2(4)(7)\cos\theta$$

so $\cos\theta = -.285714$ or $\theta = 106.60155°$. Hence, the area is

$$\frac{1}{2}(4)(7)\sin 106.60155° = 13.42\ m^2$$

One could also use Heron's Formula to get the area. Half of the perimeter, s, is $s = \frac{4+7+9}{2} = 10$. The area is therefore

$$\sqrt{10(10 - 4)(10 - 7)(10 - 9)} = \sqrt{180} = 13.42\ m^2$$