Calculus Contest 2008

1) $\lim\limits_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$

(a) 0 (b) 1/2 (c) 1 (d) 2 (e) none of these

2) $\lim\limits_{x \to \infty} \frac{\sqrt{x^2 + 4}}{x}$

(a) 0 (b) 1/2 (c) 2 (d) 4 (e) none of these

3) $\frac{d}{dx} \log_2 \left(x^2 + 1\right)$

(a) $\frac{1}{x^2 + 1}$ (b) $\frac{2x}{x^2 + 1}$ (c) $\frac{1}{\left(x^2 + 1\right) \ln 2}$

(d) $\frac{2x}{\left(x^2 + 1\right) \ln 2}$ (e) none of these

4) $\frac{d}{dx} \tan^{-1}(2x)$

(a) $\frac{1}{4x^2 + 1}$ (b) $\frac{2}{x^2 + 1}$ (c) $\frac{2x}{4x^2 + 1}$

(d) $\frac{2}{4x^2 + 1}$ (e) none of these

5) If $2x - x^2 \le f(x) \le x^2 - 2x + 2$ for all x, then $\lim\limits_{x \to 1} f(x)$ is

(a) -1 (b) 0 (c) 1 (d) 2 (e) none of these

6) $\lim\limits_{x \to -2} \frac{x^3 + 8}{x + 2}$

(a) 1 (b) 1/2 (c) 4 (d) 8 (e) none of these

7) $\dfrac{d|x|}{dx}$

(a) 1 (b) $\dfrac{|x|}{x}$ (c) $\dfrac{1}{|x|}$ (d) 0 (e) none of these

8) If $y = \dfrac{x}{x+1}$, then the equation of the tangent at $x = -2$ is

(a) $y = x + 4$ (b) $y = x + 2$ (c) $y = x - 2$

(d) $y = 2x$ (e) none of these

9) If $x^3 - xy + y^2 = x + 1$ then y' is

(a) $\dfrac{3x^2 - 1}{x - 2}$ (b) $\dfrac{3x^2 + y - 1}{x - 2y}$ (c) $\dfrac{3x^2 - y - 1}{x - 2y}$

(d) $\dfrac{3x^2 + y}{x - 2y}$ (e) none of these

10) If $f(x) = \begin{cases} x, & x \le 1 \\ 2x^2 + a, & x > 1 \end{cases}$ the value of a that makes f continuous is

(a) -1 (b) 0 (c) 1 (d) 2 (e) none of these

11) If f and g are continuous functions and
$\lim\limits_{x \to 1} f(x) = 2$, $\lim\limits_{x \to 0} g(x) = 1$, then $\lim\limits_{x \to 0} f(g(x))$

(a) 0 (b) 1 (c) 2 (d) 4 (e) none of these

12) If $f(x) = \sin 2x$, then $f^{(21)}(x)$ is

(a) $2^{21} \sin 2x$ (b) $2^{21} \cos 2x$ (c) $-2^{21} \sin 2x$ (d) $-2^{21} \sin 2x$ (e) none of these

13) If $f(x) = x^3 - 6x^2 + 9x$ then f is increasing on the interval(s)

(a) $x < 1$ (b) $1 < x < 3$ (c) $x < 1$ *and* $x > 3$ (d) $x > 3$
(e) none of these

14) If $f(x) = \dfrac{x^3}{x-1}$ then which of the following is an asymptote for f(x)?

(a) $y = x^2$ (b) $y = x^2 + x$ (c) $y = x^2 + 1$ (d) $y = x^2 + x + 1$
(e) none of these

15) If $f(x)$ is continuous on the interval [a,b] then which of the following statements are true

(a) $f(x)$ has both an absolute minimum and maximum on [a,b]
(b) $f(x)$ has only an absolute minimum on [a,b]
(c) $f(x)$ has only an absolutemaximum on [a,b]
(d) $f(x)$ has neither an absolute minimum nor maximum on [a,b]
(e) nothing can be said without knowing what $f(x)$ is

16) The area bound between $y = x^3 - x$ and the x-axis is

(a) – 1/4 (b) – 1/2 (c) 0 (d) 1/4 (e) none of these

17) $\displaystyle\int \frac{x+1}{x^2+1}\,dx$

(a) $\ln\left(x^2+1\right) + c$ (b) $\tan^{-1} x + c$ (c) $\ln\left(x^2+1\right) + \tan^{-1} x + c$
(d) $\frac{1}{2}\ln\left(x^2+1\right) + \tan^{-1} x + c$ (e) none of these

18) The position of a car after t minutes driving in a straight line is given by

$$s(t) = \tfrac{1}{2}t^2 - \tfrac{1}{12}t^3$$

The instantaneous velocity of the car at $t = 2$ is

(a) $\frac{1}{12}$ (b) $\frac{5}{12}$ (c) $\frac{1}{2}$ (d) $\frac{4}{3}$ (e) none of these

19) The average of $y = \sin x$ on $[0,\pi]$ is

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{\pi}$ (e) none of these

20) The value of A such that $y = Axe^x$ satisfies $\frac{dy}{dx} - y = 4e^x$ is

(a) 1 (b) 2 (c) 4 (d) 5 (e) none of these

21) $\frac{d}{dx} x^x$

(a) $x^x(\ln x + 1)$ (b) $x \ln x$ (c) $\ln x + 1$ (d) x^x (e) none of these

22) If $f(x) = x^3 + 2x$ and if $g(x)$ is the inverse of $f(x)$, then $g'(3)$ is

(a) $\frac{1}{29}$ (b) $\frac{1}{5}$ (c) 3 (d) 29 (e) none of these

23) If f and g are differentiable functions where
$f(1) = 3$, $g(1) = 2$, $f'(1) = 4$, $f'(2) = 3$, $g'(1) = -2$, and $g'(3) = 5$,
and h is defined as $h(x) = f(g(x))$ then $h'(1)$ is

(a) -6 (b) 3 (c) 4 (d) -2 (e) none of these

24) $\frac{d}{dx}\int_1^{x^2} \sin t^2 \, dt$

(a) $\sin x^2$ (b) $\sin x^4$ (c) $2x \sin x^2$ (d) $2x \sin x^4$ (e) none of these

25) The values a and b such that $y = 5x + 1$ is the equation of the tangent to $y = x^2 + ax + b$ at $x = 1$ are

(a) $a = 2,\ b = 3$ (b) $a = 5,\ b = 1$ (c) $a = 2,\ b = 1$ (d) $a = 5,\ b = 3$

(e) none of these

Name: ____

Calculus Tie Breakers

1) Find the value of k such that $y = \frac{k}{x}$ and $y = 2\sqrt{x}$ intersect at right angles?

2) Evaluate $\int \frac{dx}{\sqrt{2} + \sin x + \cos x}$.

3) Find the point on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.

Regional Calculus Contest – Solutions

1 – b	11 – c	21 – a
2 – e	12 – b	22 – b
3 – d	13 – c	23 – a
4 – d	14 – d	24 – d
5 – c	15 – a	25 – e
6 – e	16 – e	
7 – b	17 – d	
8 – a	18 – e	
9 – c	19 – d	
10 – a	20 – c	

Tie Breaker 1

Suppose that the two curves intersect at $x = a$ (to be determined). Then from the y value of each curve we obtain $\frac{k}{a} = 2\sqrt{a}$. Since the curves are to intersect at right angles their tangents at the point of intersect will be perpendicular. Hence, the derivatives for each are $y' = -\frac{k}{x^2}$, and $y' = \frac{1}{\sqrt{x}}$. At $x = a$, the respective slopes are $m_1 = -\frac{k}{a^2}$, $m_2 = \frac{1}{\sqrt{a}}$, from which we obtain the perpendicular condition $m_1 m_2 = -\frac{k}{a^2}.\frac{1}{\sqrt{a}} = -1$. This gives two equations for the unknowns a and k. Solving gives $a = 2$ nad $k = 4\sqrt{2}$.

Tie Breaker 2

Integrate $\int \frac{dx}{\sqrt{2} + \sin x + \cos x}$

Solution. If we use the identity $\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\sin x + \cos x\right)$, then

$$\int \frac{dx}{\sqrt{2} + \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}\int \frac{dx}{1 + \sin\left(x + \frac{\pi}{4}\right)}.$$

If we let $u = x + \frac{\pi}{4}$ then the integral becomes

$$\frac{1}{\sqrt{2}}\int\frac{du}{1+\sin(u)} = \frac{1}{\sqrt{2}}\int\frac{(1-\sin(u))\,du}{\cos^2 u} = \frac{1}{\sqrt{2}}\int \sec^2 u - \sec u\tan u \; du\,.$$

Each readily integrates giving the final solution $\frac{1}{\sqrt{2}}\left(\tan\left(x+\frac{\pi}{4}\right)-\sec\left(x+\frac{\pi}{4}\right)\right) + c$

Tie Breaker 3

If we let $x = a$ be the point of tangency, then the tangent is $y-\left(1-a^2\right) = -2a(x-a)$.

This intersects each axis $\left(0, a^2+1\right)$ and $\left(\frac{a^2+1}{2a}, 0\right)$ giving the area of the triangle as

$A = \frac{\left(a^2+1\right)^2}{4a}$. From the first derivative $A' = \frac{\left(a^2+1\right)\left(3a^2-1\right)}{4a^2}$ gives $a = \frac{1}{\sqrt{3}}$ as a

critical point and the second derivative test $\left(A'' = \frac{3a^4+1}{2a^3} > 0\right)$ gives us a mi