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Controlling the Motion of Charged Particles in a Vacuum Electromagnetic Field from Boundary

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Controlling the Motion of Charged Particles in a Vacuum Electromagnetic Field from Boundary

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Abstract

We consider the problem of driving two non-relativistic charged particles in a bounded vacuum electromagnetic field to a same location by applying electromagnetic forces through the boundary of the domain. The dynamics of the particles is modeled by Maxwell's equations coupled with the Lorentz force law and the problem is reduced to a boundary feedback control problem. Using the perturbed energy method, we design feedback controllers and prove that the particles under the designed control move to the origin exponentially. Our result may have potential applications in particle acceleration and nuclear fusion.

Key Words: Charged particle, boundary feedback, Maxwell's equations, Lorentz force law.

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1 Introduction

The motion of charged particles is fundamental to a lot of interesting physics and engineering and has had many applications, including particle accelerators and cathode-ray and X-ray tubes. The motion of a non-relativistic charged particle in a vacuum electromagnetic field can be modeled by Maxwell's equations coupled with the Lorentz force law [3]

$$\begin{aligned}
\epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} &= \nabla\times\mathbf{B}, \quad \text{in } \Omega, && (1.1)\\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}, \quad \text{in } \Omega, && (1.2)\\
\nabla\cdot\mathbf{E} &= \nabla\cdot\mathbf{B} = 0, \quad \text{in } \Omega, && (1.3)\\
\frac{d^2\mathbf{p}}{dt^2} &= \frac{q}{m}\mathbf{E}|_{\mathbf{p}} + \frac{q}{m}\frac{d\mathbf{p}}{dt}\times\mathbf{B}|_{\mathbf{p}}, && (1.4)\\
\mathbf{E}\times\mathbf{n}|_{\partial\Omega} &= \mathbf{F}_1, \quad \mathbf{B}|_{\partial\Omega} = \mathbf{F}_2, && (1.5)
\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^3$, $\partial\Omega$ is the boundary of Ω, $\mathbf{n}$ is the outward unit normal on the boundary, the vector $\mathbf{E}$ is the electric intensity, $\mathbf{B}$ is the magnetic intensity, q is the charge, $\mathbf{p}$ is the position vector of the charge, ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space, m is the mass of the charge, $\nabla\times$ is the curl operator, $\nabla\cdot$ is the divergence operator, and $\mathbf{F}_1, \mathbf{F}_2$ are boundary feedback controllers to be designed to drive the charge $\mathbf{p}$ to a desired position such as the origin. We have used the Lorentz force law in (1.4). For two particles, the equation (1.4) needs to be replaced by the following equations

$$\begin{aligned}
\frac{d^2\mathbf{p}_1}{dt^2} &= \frac{q_1}{m_1}\mathbf{E}|_{\mathbf{p_1}} + \frac{q_1}{m_1}\frac{d\mathbf{p}_1}{dt}\times\mathbf{B}|_{\mathbf{p_1}} + \frac{1}{4\pi\varepsilon_0 m_1}\frac{q_1 q_2}{|\mathbf{p}_2-\mathbf{p}_1|^3}(\mathbf{p}_2-\mathbf{p}_1), && (1.6)\\
\frac{d^2\mathbf{p}_2}{dt^2} &= \frac{q_2}{m_2}\mathbf{E}|_{\mathbf{p_2}} + \frac{q_2}{m_2}\frac{d\mathbf{p}_2}{dt}\times\mathbf{B}|_{\mathbf{p_2}} - \frac{1}{4\pi\varepsilon_0 m_2}\frac{q_1 q_2}{|\mathbf{p}_2-\mathbf{p}_1|^3}(\mathbf{p}_2-\mathbf{p}_1), && (1.7)
\end{aligned}$$

where q_1, q_2 are the charges, $\mathbf{p}_1, \mathbf{p}_1$ are the position vectors of the particles with charges q_1, q_2, respectively, m_1, m_2 are the masses of the particles, respectively. In (1.6) and (1.7), we include a Coulomb force interaction between the particles. We assume that the particles exert no magnetic forces on each other. This means our model is only accurate in the non-relativistic regime. The case of two particles can be extended to multiple particles. Since the extension has no essential differences, we do not discuss it here.

Our goal is to design feedback controllers $\mathbf{F}_1, \mathbf{F}_2$ to drive the particle $\mathbf{p}$ to the origin in the case of one particle, and to drive two particles $\mathbf{p}_1$ and $\mathbf{p}_2$ to the origin in the case of two particles. In the latter case, if these two particles get extremely close, the equations (1.6) and (1.7) are no longer accurate since at distances similar to the size of the charge, the strong force dominates. In this case, a term modeling the strong force should be included. However, since the particles attract each other via the strong force, the equations (1.6) and (1.7) suffice to cover the effect of the strong force in driving two particles to the same location.

Control problems of Maxwell's equations have been intensively studied since Russell's pioneer work [29] in the early 1980s, when Russell was interested in control problems associated with Maxwell's equations in connection with nuclear fusion applications. Russell's results were then extended by Kime [10] to a spherical region. Using the Hilbert uniqueness method, Lagnese [18] established the boundary controllability in a general convex region. Later on, the internal controllability and controllability for the equations with variable permittivity and permeability were studied by many authors (see, e.g., [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 20, 21, 22, 23, 24, 25, 26, 27, 28, 31, 32]) via various methods such as the Lions' Hilbert uniqueness method, Russell's "controllability via stabilizability" principle, and microlocal analysis.

To our knowledge, the feedback control of (1.1)-(1.5) (or with (1.6)-(1.7)) has not been touched. In this paper we fill up this empty. Using the perturbed energy method, we design feedback controllers and prove that the designed controllers drive the particles to the origin.

2 Results

Define the energy function W_1 by

$$W_1(t) = \frac{1}{2}\int_\Omega \left(\epsilon_0|\mathbf{E}(t,x,y,z)|^2 + \frac{1}{\mu_0}|\mathbf{B}(t,x,y,z)|^2\right) dV + \frac{1}{2}\left(k|\mathbf{p}(t)|^2 + m|\mathbf{p}'(t)|^2\right),$$

where k is a positive constant. Let S_1 and S_2 be the subsets of the boundary $\partial\Omega$ such that $S_1 \bigcup S_2 = \partial\Omega$, $\mathrm{mes}(S_1) \neq 0$, and $\mathrm{mes}(S_2) \neq 0$ (mes denotes the measure). A direct calculation gives

$$\begin{aligned} W_1'(t) &= \int_\Omega \left(\epsilon_0\mathbf{E}(t)\cdot\mathbf{E}'(t) + \frac{1}{\mu_0}\mathbf{B}(t)\cdot\mathbf{B}'(t)\right) dV + k\mathbf{p}(t)\cdot\mathbf{p}'(t) + m\mathbf{p}'(t)\cdot\mathbf{p}''(t) \\ &= \int_\Omega \left(\frac{1}{\mu_0}\mathbf{E}(t)\cdot(\nabla\times\mathbf{B}) - \frac{1}{\mu_0}\mathbf{B}(t)\cdot(\nabla\times\mathbf{E})\right) dV \\ &\quad + k\mathbf{p}(t)\cdot\mathbf{p}'(t) + \mathbf{p}'(t)\cdot\left(q\mathbf{E}|_{\mathbf{p}} + q\frac{d\mathbf{p}}{dt}\times\mathbf{B}|_{\mathbf{p}}\right) \\ &= \frac{1}{\mu_0}\int_{\partial\Omega}(\mathbf{E}(t)\times\mathbf{n})\cdot\mathbf{B}(t)dS + k\mathbf{p}(t)\cdot\mathbf{p}'(t) + q\mathbf{p}'(t)\cdot\mathbf{E}|_{\mathbf{p}}. \end{aligned}$$

This motivates us to take the feedback controllers

$$
\begin{aligned}
\mathbf{B}|_{S_1} &= k\mathbf{p}(t) + q\mathbf{E}|_{\mathbf{p}} + c_3\mathbf{p}'(t), && (2.1)\\
\mathbf{E}\times\mathbf{n}|_{S_1} &= -\frac{c_1\mu_0}{\mathrm{mes}(S_1)}\mathbf{B}|_{S_1} - \frac{\mu_0}{\mathrm{mes}(S_1)}\mathbf{p}'(t), && (2.2)\\
\mathbf{B}|_{S_2} &= \mathbf{p}(t), && (2.3)\\
\mathbf{E}\times\mathbf{n}|_{S_2} &= -\frac{c_2\mu_0}{\mathrm{mes}(S_2)}\mathbf{B}|_{S_2}, && (2.4)
\end{aligned}
$$

such that

$$
W_1'(t) = -\left(c_1|k\mathbf{p}(t) + q\mathbf{E}|_{\mathbf{p}} + c_3\mathbf{p}'(t)|^2 + c_2|\mathbf{p}(t)|^2 + c_3|\mathbf{p}'(t)|^2\right), \tag{2.5}
$$

where c_1, c_2, c_3 are positive constants.

Theorem 2.1. *Assume that the domain Ω is convex. Then there exist positive constants ω, M such that the solution of the system (1.1)-(1.5) with the feedback controllers (2.1)-(2.4) satisfies*

$$
W_1(t) \le M W_1(0) e^{-\omega t}. \tag{2.6}
$$

For the case of two particles, we define the energy function by

$$
\begin{aligned}
W_2(t) &= \frac{1}{2}\int_\Omega \left(\epsilon_0|\mathbf{E}(t,x,y,z)|^2 + \frac{1}{\mu_0}|\mathbf{B}(t,x,y,z)|^2\right) dV + \frac{1}{2}\left(k_1|\mathbf{p}_1(t)|^2 + m_1|\mathbf{p}_1'(t)|^2\right)\\
&+ \frac{1}{2}\left(k_2|\mathbf{p}_2(t)|^2 + m_2|\mathbf{p}_2'(t)|^2\right),
\end{aligned}
$$

where k_1 and k_2 are a positive constants. Let S_1, S_2, S_3 and S_4 be the subsets of the boundary $\partial\Omega$ such that $\bigcup_{i=1}^4 S_i = \partial\Omega$ and $\mathrm{mes}(S_i) \neq 0$ $(i = 1,2,3,4)$. As in the case of one particle, a direct calculation gives

$$
\begin{aligned}
W_2'(t) &= \frac{1}{\mu_0}\int_{\partial\Omega} (\mathbf{E}(t)\times\mathbf{n})\cdot\mathbf{B}(t)dS\\
&+k_1\mathbf{p}_1(t)\cdot\mathbf{p}_1'(t) + \mathbf{p}_1'(t)\cdot\left(q_1\mathbf{E}|_{\mathbf{p}_1} + \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\mathbf{p}_2-\mathbf{p}_1|^3}(\mathbf{p}_2-\mathbf{p}_1)\right)\\
&+k_2\mathbf{p}_2(t)\cdot\mathbf{p}_2'(t) + \mathbf{p}_2'(t)\cdot\left(q_2\mathbf{E}|_{\mathbf{p}_2} - \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\mathbf{p}_2-\mathbf{p}_1|^3}(\mathbf{p}_2-\mathbf{p}_1)\right).
\end{aligned}
$$

Thus we derive the following boundary controllers

$$\mathbf{B}|_{S_1} = k_1\mathbf{p}_1(t) + q_1\mathbf{E}|_{\mathbf{p}_1} + c_3\mathbf{p}_1'(t) + \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\mathbf{p}_2-\mathbf{p}_1|^3}(\mathbf{p}_2-\mathbf{p}_1), \tag{2.7}$$

$$\mathbf{E}\times\mathbf{n}|_{S_1} = -\frac{c_1\mu_0}{\mathrm{mes}(S_1)}\mathbf{B}|_{S_1} - \frac{\mu_0}{\mathrm{mes}(S_1)}\mathbf{p}_1'(t), \tag{2.8}$$

$$\mathbf{B}|_{S_2} = \mathbf{p}_1(t), \tag{2.9}$$

$$\mathbf{E}\times\mathbf{n}|_{S_2} = -\frac{c_2\mu_0}{\mathrm{mes}(S_2)}\mathbf{B}|_{S_2}, \tag{2.10}$$

$$\mathbf{B}|_{S_3} = k_2\mathbf{p}_2(t) + q_2\mathbf{E}|_{\mathbf{p}_2} + c_6\mathbf{p}_2'(t) - \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\mathbf{p}_2-\mathbf{p}_1|^3}(\mathbf{p}_2-\mathbf{p}_1), \tag{2.11}$$

$$\mathbf{E}\times\mathbf{n}|_{S_3} = -\frac{c_4\mu_0}{\mathrm{mes}(S_3)}\mathbf{B}|_{S_3} - \frac{\mu_0}{\mathrm{mes}(S_3)}\mathbf{p}_2'(t), \tag{2.12}$$

$$\mathbf{B}|_{S_4} = \mathbf{p}_2(t), \tag{2.13}$$

$$\mathbf{E}\times\mathbf{n}|_{S_4} = -\frac{c_5\mu_0}{\mathrm{mes}(S_4)}\mathbf{B}|_{S_4} \tag{2.14}$$

where c's are positive constants. With these feedback controllers we have

$$W_2'(t) = -\left(c_1|\mathbf{B}|_{S_1}(t)|^2 + c_2|\mathbf{p}_1(t)|^2 + c_3|\mathbf{p}_1'(t)|^2 + c_4|\mathbf{B}|_{S_3}(t)|^2 + c_5|\mathbf{p}_2(t)|^2 + c_6|\mathbf{p}_2'(t)|^2\right). \tag{2.15}$$

Theorem 2.2. *Assume that the domain Ω is convex. Then there exist positive constants ω, M such that the solution of the system (1.1), (1.2), (1.3), (1.6), (1.7) with the feedback controllers (2.7)-(2.14) satisfies*

$$W_2(t) \le MW_2(0)e^{-\omega t}. \tag{2.16}$$

3 Proofs

The following identity is a small variant of the identity developed in [4] and plays a key rule in dealing with Maxwell's equations. For reader's convenience, we provide a concise proof by using tensors.

Lemma 3.1. *Let $\mathbf{E}, \mathbf{B}$ satisfy the Maxwell's equations (1.1) and (1.2). Then for any differ-*

entiable vector function $\mathbf{v} = (v_1, v_2, v_3)$ *and any constant* δ*, we have*

$$\begin{aligned}
&\mu_0 \frac{\partial}{\partial t}\Big(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2 + 2\delta\frac{\epsilon_0}{\mu_0}\mathbf{v}\cdot(\mathbf{B}\times\mathbf{E})\Big) \\
= \;& \nabla\cdot\Big(2(\mathbf{B}\times\mathbf{E}) + \delta\left(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right)\mathbf{v} - 2\delta\epsilon_0(\mathbf{E}\cdot\mathbf{v})\mathbf{E} - 2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})\mathbf{B}\Big) \\
&+2\delta\sum_{i,j=1}^{3}\frac{\partial v_i}{\partial x_j}\left(\epsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j\right) - \delta\nabla\cdot\mathbf{v}(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2) \\
&+2\delta\epsilon_0(\mathbf{E}\cdot\mathbf{v})(\nabla\cdot\mathbf{E}) + 2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\nabla\cdot\mathbf{B}).
\end{aligned} \tag{3.1}$$

Proof. In this proof, Einstein summation convention is assumed. We also raise and lower indices without abandon because our problem is set in flat space. Using the tensor notation, we can write $\nabla\cdot\mathbf{E} = \partial_\mu E^\mu$ and $\nabla\times\mathbf{E} = \epsilon^{\mu\alpha\beta}\partial_\alpha B_\beta$ and then (1.1) and (1.2) can be rewritten as $\dot{E}^\mu = \frac{1}{\epsilon_0\mu_0}\epsilon^{\mu\alpha\beta}\partial_\alpha B_\beta$ and $\dot{B}^\mu = -\epsilon^{\mu\alpha\beta}\partial_\alpha E_\beta$, respectively, where

$$\epsilon^{\mu\alpha\beta} = \begin{cases} 1, & \text{if } \mu\alpha\beta \text{ is an even permutation of 123,} \\ -1, & \text{if } \mu\alpha\beta \text{ is an odd permutation of 123,} \\ 0, & \text{otherwise.} \end{cases}$$

It then follows that

$$\begin{aligned}
&\mu_0\frac{\partial}{\partial t}\Big(\epsilon_0 E_\mu E^\mu + \frac{1}{\mu_0}B_\mu B^\mu + 2\delta\frac{\epsilon_0}{\mu_0}v_\mu\epsilon^{\mu\alpha\beta}B_\alpha E_\beta\Big) \\
= \;& 2\mu_0\epsilon_0\dot{E}_\mu E^\mu + 2\dot{B}_\mu B^\mu + 2\delta\epsilon_0 v_\mu\epsilon^{\mu\alpha\beta}[\dot{B}_\alpha E_\beta + B_\alpha\dot{E}_\beta] \\
= \;& 2\epsilon_\mu{}^{\alpha\beta}(\partial_\alpha B_\beta)E^\mu - 2\epsilon_\mu{}^{\alpha\beta}(\partial_\alpha E_\beta)B^\mu + 2\delta\epsilon_0 v_\mu\epsilon^{\mu\alpha\beta}\left[-\epsilon_\alpha{}^{\gamma\delta}(\partial_\gamma E_\delta)E_\beta + \frac{1}{\epsilon_0\mu_0}\epsilon_\beta{}^{\gamma\delta}(\partial_\gamma B_\delta)B_\alpha\right].
\end{aligned}$$

After reindexing and factoring, the first two terms can be rewritten as the derivative of a product, and then we obtain

$$\begin{aligned}
&\mu_0\frac{\partial}{\partial t}\Big(\epsilon_0 E_\mu E^\mu + \frac{1}{\mu_0}B_\mu B^\mu + 2\delta\frac{\epsilon_0}{\mu_0}v_\mu\epsilon^{\mu\alpha\beta}B_\alpha E_\beta\Big) \\
= \;& 2\partial_\alpha(\epsilon^{\alpha\beta\mu}B_\beta E_\mu) - 2\delta\epsilon_0 v_\mu\epsilon^{\mu\alpha\beta}\epsilon_{\alpha\gamma\delta}(\partial^\gamma E^\delta)E_\beta + 2\frac{\delta}{\mu_0}v_\mu\epsilon^{\mu\alpha\beta}\epsilon_{\beta\gamma\delta}(\partial^\gamma B^\delta)B_\alpha.
\end{aligned} \tag{3.2}$$

Using the following identity

$$\epsilon_{\mu\alpha\beta}\epsilon^{ijk} = \delta^{ijk}_{\mu\alpha\beta},$$

we obtain

$$\begin{aligned}
\epsilon^{\mu\alpha\beta}\epsilon_{\alpha\gamma\delta} &= -\epsilon^{\alpha\mu\beta}\epsilon_{\alpha\gamma\delta} = \delta^\mu_\delta\delta^\beta_\gamma - \delta^\mu_\gamma\delta^\beta_\delta, \\
\epsilon^{\mu\alpha\beta}\epsilon_{\beta\gamma\delta} &= \epsilon^{\beta\mu\alpha}\epsilon_{\beta\gamma\delta} = \delta^\mu_\gamma\delta^\alpha_\delta - \delta^\mu_\delta\delta^\alpha_\gamma,
\end{aligned}$$

where

$$\delta_\alpha^\beta = \left\{ \begin{array}{ll} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{array} \right. \quad , \quad \delta_{\mu\alpha\beta}^{ijk} = \left| \begin{array}{ccc} \delta_\mu^i & \delta_\alpha^i & \delta_\beta^i \\ \delta_\mu^j & \delta_\alpha^j & \delta_\beta^j \\ \delta_\mu^k & \delta_\alpha^k & \delta_\beta^k \end{array} \right|.$$

Substituting these equations into (3.2), we get

$$\begin{aligned}
& \mu_0 \frac{\partial}{\partial t}\Big(\epsilon_0 E_\mu E^\mu + \frac{1}{\mu_0} B_\mu B^\mu + 2\delta \frac{\epsilon_0}{\mu_0} v_\mu \epsilon^{\mu\alpha\beta} B_\alpha E_\beta\Big) \\
= \;& 2\partial_\alpha(\epsilon^{\alpha\beta\mu} B_\beta E_\mu) - 2\delta\epsilon_0 v_\mu [\delta_\delta^\mu \delta_\gamma^\beta - \delta_\gamma^\mu \delta_\delta^\beta](\partial^\gamma E^\delta) E_\beta + 2\frac{\delta}{\mu_0} v_\mu [\delta_\gamma^\mu \delta_\delta^\alpha - \delta_\delta^\mu \delta_\gamma^\alpha](\partial^\gamma B^\delta) B_\alpha \\
= \;& 2\partial_\alpha(\epsilon^{\alpha\beta\mu} B_\beta E_\mu) - 2\delta\epsilon_0 v_\mu (\partial^\beta E^\mu) E_\beta + 2\delta\epsilon_0 v_\mu (\partial^\mu E^\beta) E_\beta \\
& + 2\frac{\delta}{\mu_0} v_\mu (\partial^\mu B^\alpha) B_\alpha - 2\frac{\delta}{\mu_0} v_\mu (\partial^\alpha B^\mu) B_\alpha \\
= \;& 2\partial_\alpha(\epsilon^{\alpha\beta\mu} B_\beta E_\mu) - 2\delta\epsilon_0 \partial^\beta (v_\mu E^\mu) E_\beta + 2\delta\epsilon_0 \partial^\beta (v_\mu) E^\mu E_\beta + \delta\epsilon_0 v_\mu \partial^\mu (E^\beta E_\beta) \\
& + \frac{\delta}{\mu_0} v_\mu \partial^\mu (B^\alpha B_\alpha) - 2\frac{\delta}{\mu_0} \partial^\alpha (v_\mu B^\mu) B_\alpha + 2\frac{\delta}{\mu_0} \partial^\alpha (v_\mu) B^\mu B_\alpha \\
= \;& 2\partial_\alpha(\epsilon^{\alpha\beta\mu} B_\beta E_\mu) + 2\delta(\partial_\alpha v_\mu) \left[\epsilon_0 E^\alpha E^\mu + \frac{1}{\mu_0} B^\alpha B^\mu\right] - 2\delta\epsilon_0 \partial^\beta (v_\mu E^\mu) E_\beta \\
& - 2\frac{\delta}{\mu_0} \partial^\alpha (v_\mu B^\mu) B_\alpha + \delta\epsilon_0 v_\mu \partial^\mu (E^\beta E_\beta) + \frac{\delta}{\mu_0} v_\mu \partial^\mu (B^\alpha B_\alpha) \\
= \;& 2\partial_\alpha(\epsilon^{\alpha\beta\mu} B_\beta E_\mu) + 2\delta(\partial_\alpha v_\mu) \left[\epsilon_0 E^\alpha E^\mu + \frac{1}{\mu_0} B^\alpha B^\mu\right] - 2\delta\epsilon_0 \partial^\beta (v_\mu E^\mu E_\beta) \\
& + 2\delta\epsilon_0 (v_\mu E^\mu)(\partial^\beta E_\beta) - 2\frac{\delta}{\mu_0} \partial^\alpha (v_\mu B^\mu B_\alpha) + 2\frac{\delta}{\mu_0} (v_\mu B^\mu)(\partial^\alpha B_\alpha) + \delta\epsilon_0 \partial^\mu (v_\mu E^\beta E_\beta) \\
& + \frac{\delta}{\mu_0} \partial^\mu (v_\mu B^\alpha B_\alpha) - (\delta \partial^\mu v_\mu) \left(\epsilon_0 (E^\beta E_\beta) + \frac{1}{\mu_0} (B^\alpha B_\alpha)\right)
\end{aligned}$$

which is precisely the identity (3.1). □

We are now ready to use the perturbed energy method to prove our main results. We prove Theorem 2.2. The proof of Theorem 2.1 is virtually identical by taking $q_2 = 0$ in the following proof.

Proof of Theorem 2.2. We define the perturbed energy by

$$\begin{aligned}
W_\delta(t) &= W_2(t) + \delta \int_\Omega \frac{\epsilon_0}{\mu_0} \mathbf{v} \cdot (\mathbf{B} \times \mathbf{E}) dV \\
&= \frac{1}{2} \int_\Omega \left(\epsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0} |\mathbf{B}|^2 + 2\delta \frac{\epsilon_0}{\mu_0} \mathbf{v} \cdot (\mathbf{B} \times \mathbf{E})\right) dV \\
&+ \frac{1}{2} \left(k_1 |\mathbf{p}_1|^2 + m_1 |\mathbf{p}_1'|^2\right) + \frac{1}{2} \left(k_2 |\mathbf{p}_2|^2 + m_2 |\mathbf{p}_2'|^2\right),
\end{aligned}$$

The theorem will be proved if we can show

$$W'_\delta(t) \le -\omega W_\delta(t) \tag{3.3}$$

for some positive constant ω. Let $\mathbf{x}_0 \in \Omega$. Taking $\mathbf{v} = \mathbf{x} - \mathbf{x}_0$ in the identity (3.1) and using Stoke's Theorem, it follows from (3.1) that

$$\begin{aligned}
W'_\delta(t) &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2 + 2\delta\frac{\epsilon_0}{\mu_0}\mathbf{v}\cdot(\mathbf{B}\times\mathbf{E})\right) dV \\
&\quad + (k_1\mathbf{p}_1\cdot\mathbf{p}'_1 + m_1\mathbf{p}'_1\cdot\mathbf{p}''_1) + (k_2\mathbf{p}_2\cdot\mathbf{p}'_2 + m_2\mathbf{p}'_2\cdot\mathbf{p}''_2) \\
&= \frac{1}{2\mu_0}\int_{\partial\Omega}\left(2\mathbf{B}\cdot[\mathbf{E}\times\mathbf{n}] + \frac{\delta}{\mu_0}|\mathbf{B}|^2(\mathbf{v}\cdot\mathbf{n}) - 2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right) dS \\
&\quad + \frac{1}{2\mu_0}\int_{\partial\Omega}\big(2\delta\epsilon_0[(\mathbf{E}\cdot\mathbf{E})(\mathbf{v}\cdot\mathbf{n}) - (\mathbf{E}\cdot\mathbf{v})(\mathbf{E}\cdot\mathbf{n})] - \delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\big) dS \\
&\quad - \frac{\delta}{\mu_0}\int_\Omega\left(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right) dV + (k_1\mathbf{p}_1\cdot\mathbf{p}'_1 + m_1\mathbf{p}'_1\cdot\mathbf{p}''_1) \\
&\quad + (k_2\mathbf{p}_2\cdot\mathbf{p}'_2 + m_2\mathbf{p}'_2\cdot\mathbf{p}''_2)\,.
\end{aligned}$$

Here we have added and subtracted $\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})$ inside the surface integral. Using the identity $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$ and the equations (1.6) and (1.7), we then obtain

$$\begin{aligned}
W'_\delta(t) &= \frac{1}{2\mu_0}\int_{\partial\Omega}\left(2\mathbf{B}\cdot[\mathbf{E}\times\mathbf{n}] + \frac{\delta}{\mu_0}|\mathbf{B}|^2(\mathbf{v}\cdot\mathbf{n}) - 2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right) dS \\
&\quad + \frac{1}{2\mu_0}\int_{\partial\Omega}\big(2\delta\epsilon_0[(\mathbf{E}\times\mathbf{n})\cdot(\mathbf{E}\times\mathbf{v})] - \delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\big) dS - \frac{\delta}{\mu_0}\int_\Omega\left(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right) dV \\
&\quad + k_1\mathbf{p}_1\cdot\mathbf{p}'_1 + q_1\mathbf{p}'_1\cdot\mathbf{E}_{p_1} + \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{p_2}-\mathbf{p_1}|^3}\mathbf{p}'_1\cdot(\mathbf{p_2}-\mathbf{p_1}) \\
&\quad + k_2\mathbf{p}_2\cdot\mathbf{p}'_2 + q_2\mathbf{p}'_2\cdot\mathbf{E}_{p_2} - \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{p_2}-\mathbf{p_1}|^3}\mathbf{p}'_2\cdot(\mathbf{p_2}-\mathbf{p_1}).
\end{aligned}$$

Splitting the surface integral on $\partial\Omega$ into four surface integrals on S_1, S_2, S_3 and S_4 and using the feedback controllers (2.7)-(2.14), we deduce that

$$
\begin{aligned}
W'_\delta(t) \;=\;& \frac{1}{2\mu_0}\int_{S_1}\left[|\mathbf{B}|^2\left(-\frac{2c_1\mu_0}{\mathrm{mes}(S_1)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-\frac{2\mu_0}{\mathrm{mes}(S_1)}\mathbf{B}\cdot\mathbf{p'}_1-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right]dS\\
&+\frac{1}{2\mu_0}\int_{S_1}\left(-\frac{2\delta\epsilon_0\mu_0}{\mathrm{mes}(S_1)}[(c_1\mathbf{B}+\mathbf{p'}_1)\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_2}\left(|\mathbf{B}|^2\left(-\frac{2c_2\mu_0}{\mathrm{mes}(S_2)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_2}\left(-\frac{2\delta c_2\epsilon_0\mu_0}{\mathrm{mes}(S_2)}[\mathbf{B}\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_3}\left[|\mathbf{B}|^2\left(-\frac{2c_4\mu_0}{\mathrm{mes}(S_3)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-\frac{2\mu_0}{\mathrm{mes}(S_3)}\mathbf{B}\cdot\mathbf{p'}_2-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right]dS\\
&+\frac{1}{2\mu_0}\int_{S_3}\left(-\frac{2\delta\epsilon_0\mu_0}{\mathrm{mes}(S_3)}[(c_4\mathbf{B}+\mathbf{p'}_2)\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_4}\left(|\mathbf{B}|^2\left(-\frac{2c_5\mu_0}{\mathrm{mes}(S_4)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_4}\left(-\frac{2\delta c_5\epsilon_0\mu_0}{\mathrm{mes}(S_4)}[\mathbf{B}\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS-\frac{\delta}{\mu_0}\int_\Omega\left(\epsilon_0|\mathbf{E}|^2+\frac{1}{\mu_0}|\mathbf{B}|^2\right)dV\\
&+k_1\mathbf{p}_1\cdot\mathbf{p}'_1+q_1\mathbf{p}'_1\cdot\mathbf{E}_{p_1}+\frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{p_2}-\mathbf{p_1}|^3}\mathbf{p}'_1\cdot(\mathbf{p_2}-\mathbf{p_1})\\
&+k_2\mathbf{p}_2\cdot\mathbf{p}'_2+q_2\mathbf{p}'_2\cdot\mathbf{E}_{p_2}-\frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{p_2}-\mathbf{p_1}|^3}\mathbf{p}'_2\cdot(\mathbf{p_2}-\mathbf{p_1}).
\end{aligned}
\tag{3.4}
$$

Using the controllers (2.7) and (2.11), we obtain

$$
\begin{aligned}
\frac{1}{2\mu_0}\int_{S_1}-\frac{2\mu_0}{\mathrm{mes}(S_1)}\mathbf{B}\cdot\mathbf{p'}_1dS \;=\;& -c_3|\mathbf{p}'_1(t)|^2-k_1\mathbf{p}_1\cdot\mathbf{p}'_1-q_1\mathbf{p}'_1\cdot\mathbf{E}_{p_1}\\
&-\frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{p_2}-\mathbf{p_1}|^3}\mathbf{p}'_1\cdot(\mathbf{p_2}-\mathbf{p_1}),\\
\frac{1}{2\mu_0}\int_{S_3}-\frac{2\mu_0}{\mathrm{mes}(S_3)}\mathbf{B}\cdot\mathbf{p'}_2dS \;=\;& -c_6|\mathbf{p}'_2(t)|^2-k_1\mathbf{p}_2\cdot\mathbf{p}'_2-q_2\mathbf{p}'_2\cdot\mathbf{E}_{p_2}\\
&+\frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{p_2}-\mathbf{p_1}|^3}\mathbf{p}'_2\cdot(\mathbf{p_2}-\mathbf{p_1}).
\end{aligned}
$$

It then follows from (3.4) that

$$
\begin{aligned}
W'_\delta(t) \;=\;& \frac{1}{2\mu_0}\int_{S_1}\left[|\mathbf{B}|^2\left(-\frac{2c_1\mu_0}{\mathrm{mes}(S_1)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right]dS\\
&+\frac{1}{2\mu_0}\int_{S_1}\left(-\frac{2\delta\epsilon_0\mu_0}{\mathrm{mes}(S_1)}[(c_1\mathbf{B}+\mathbf{p'}_1)\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_2}\left(|\mathbf{B}|^2\left(-\frac{2c_2\mu_0}{\mathrm{mes}(S_2)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_2}\left(-\frac{2\delta c_2\epsilon_0\mu_0}{\mathrm{mes}(S_2)}[\mathbf{B}\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_3}\left[|\mathbf{B}|^2\left(-\frac{2c_4\mu_0}{\mathrm{mes}(S_3)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right]dS\\
&+\frac{1}{2\mu_0}\int_{S_3}\left(-\frac{2\delta\epsilon_0\mu_0}{\mathrm{mes}(S_3)}[(c_4\mathbf{B}+\mathbf{p'}_2)\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_4}\left(|\mathbf{B}|^2\left(-\frac{2c_5\mu_0}{\mathrm{mes}(S_4)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_4}\left(-\frac{2\delta c_5\epsilon_0\mu_0}{\mathrm{mes}(S_4)}[\mathbf{B}\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
&-\frac{\delta}{\mu_0}\int_{\Omega}\left(\epsilon_0|\mathbf{E}|^2+\frac{1}{\mu_0}|\mathbf{B}|^2\right)dV-c_3|\mathbf{p}'_1(t)|^2-c_6|\mathbf{p}'_2(t)|^2. \qquad (3.5)
\end{aligned}
$$

We now estimate each surface integral. Let

$$M=\max_{\mathbf{x}\in\partial\Omega}|\mathbf{v}(\mathbf{x})|=\max_{\mathbf{x}\in\partial\Omega}|\mathbf{x}-\mathbf{x}_0|.$$

The convexity of the domain Ω implies that there exists a positive constant γ such that

$$\min_{\mathbf{x}\in\partial\Omega}\mathbf{v}\cdot\mathbf{n}=\min_{\mathbf{x}\in\partial\Omega}(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}=\gamma>0.$$

Using Young's inequality, we deduce that

$$\left|\frac{2\delta\epsilon_0\mu_0}{\mathrm{mes}(S_1)}[(c_1\mathbf{B}+\mathbf{p'}_1)\cdot(\mathbf{E}\times\mathbf{v})]\right|\le C\delta(|\mathbf{B}|^2+|\mathbf{p'}_1|^2)+\frac{1}{2}\delta\epsilon_0\gamma|\mathbf{E}|^2.$$

Hereafter C denotes a generic positive constant that may change from line to line and that depends on many parameters such as $\epsilon_0,\mu_0,\mathrm{mes}(S_1),\gamma,M$, but independent of δ. Using these estimates, we estimate the surface integral on S_1 as follows

$$
\begin{aligned}
&\frac{1}{2\mu_0}\int_{S_1}\left[|\mathbf{B}|^2\left(-\frac{2c_1\mu_0}{\mathrm{mes}(S_1)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right]dS\\
&+\frac{1}{2\mu_0}\int_{S_1}\left(-\frac{2\delta\epsilon_0\mu_0}{\mathrm{mes}(S_1)}[(c_1\mathbf{B}+\mathbf{p'}_1)\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
\le\;&\frac{1}{2\mu_0}\int_{S_1}\left[|\mathbf{B}|^2\left(-\frac{2c_1\mu_0}{\mathrm{mes}(S_1)}+\frac{\delta M}{\mu_0}\right)+2\frac{\delta M}{\mu_0}|\mathbf{B}|^2\right]dS\\
&+\frac{1}{2\mu_0}\int_{S_1}\left(C\delta(|\mathbf{B}|^2+|\mathbf{p'}_1|^2)+\frac{1}{2}\delta\epsilon_0\gamma|\mathbf{E}|^2-\delta\epsilon_0\gamma|\mathbf{E}|^2\right)dS\\
\le\;&\frac{1}{2\mu_0}\int_{S_1}|\mathbf{B}|^2\left(-\frac{2c_1\mu_0}{\mathrm{mes}(S_1)}+\delta C\right)dS+C\delta|\mathbf{p'}_1|^2\\
\le\;&C\delta|\mathbf{p'}_1|^2 \qquad (3.6)
\end{aligned}
$$

for sufficiently small δ. In the same way, we can estimate all other surface integrals as follows

$$
\begin{aligned}
&\frac{1}{2\mu_0}\int_{S_2}\left(|\mathbf{B}|^2(-\frac{2c_2\mu_0}{\mathrm{mes}(S_2)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n}))-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_2}\left(-\frac{2\delta c_2\epsilon_0\mu_0}{\mathrm{mes}(S_2)}[\mathbf{B}\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
\le\;&\frac{1}{2\mu_0}\int_{S_2}|\mathbf{B}|^2\left(-\frac{2c_2\mu_0}{\mathrm{mes}(S_2)}+\delta C\right)dS\quad(\text{use (2.9)})\\
\le\;&-\frac{c_2}{2}|\mathbf{p}_1|^2, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2\mu_0}\int_{S_3}\left[|\mathbf{B}|^2\left(-\frac{2c_4\mu_0}{\mathrm{mes}(S_3)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n})\right)-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right]dS\\
&+\frac{1}{2\mu_0}\int_{S_3}\left(-\frac{2\delta\epsilon_0\mu_0}{\mathrm{mes}(S_3)}[(c_4\mathbf{B}+\mathbf{p'}_2)\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
\le\;&\frac{1}{2\mu_0}\int_{S_3}|\mathbf{B}|^2\left(-\frac{2c_4\mu_0}{\mathrm{mes}(S_3)}+\delta C\right)dS+C\delta|\mathbf{p'}_2|^2\\
\le\;&C\delta|\mathbf{p'}_2|^2, \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2\mu_0}\int_{S_4}\left(|\mathbf{B}|^2(-\frac{2c_5\mu_0}{\mathrm{mes}(S_4)}+\frac{\delta}{\mu_0}(\mathbf{v}\cdot\mathbf{n}))-2\frac{\delta}{\mu_0}(\mathbf{B}\cdot\mathbf{v})(\mathbf{B}\cdot\mathbf{n})\right)dS\\
&+\frac{1}{2\mu_0}\int_{S_4}\left(-\frac{2\delta c_5\epsilon_0\mu_0}{\mathrm{mes}(S_4)}[\mathbf{B}\cdot(\mathbf{E}\times\mathbf{v})]-\delta\epsilon_0|\mathbf{E}|^2(\mathbf{v}\cdot\mathbf{n})\right)dS\\
\le\;&\frac{1}{2\mu_0}\int_{S_4}|\mathbf{B}|^2\left(-\frac{2c_5\mu_0}{\mathrm{mes}(S_4)}+\delta C\right)dS\quad(\text{use (2.13)})\\
\le\;&-\frac{c_5}{2}|\mathbf{p}_2|^2 \qquad (3.9)
\end{aligned}
$$

for sufficiently small δ. It then follows from (3.5)-(3.9) that

$$\begin{aligned} W_\delta'(t) &\le -\frac{\delta}{\mu_0}\int_\Omega \left(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right) dV - c_3|\mathbf{p}_1'(t)|^2 - c_6|\mathbf{p}_2'(t)|^2 \\ &\quad -\frac{c_2}{2}|\mathbf{p}_1|^2 - \frac{c_5}{2}|\mathbf{p}_2|^2 + C\delta(|\mathbf{p}_1'(t)|^2 + |\mathbf{p}_2'(t)|^2) \\ &\le -\frac{\delta}{\mu_0}\int_\Omega \left(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right) dV - \frac{1}{2}(c_2|\mathbf{p}_1|^2 + c_3|\mathbf{p}_1'(t)|^2 + c_5|\mathbf{p}_2|^2 + c_6|\mathbf{p}_2'(t)|^2) \end{aligned} \tag{3.10}$$

Moreover, we can readily show that there are positive constants C_1 and C_2 such that

$$C_1 W \le W_\delta \le C_2 W$$

for sufficiently small δ. It then follows from (3.10) that there exists a positive constant $\omega = \omega(\delta)$ such that

$$W_\delta'(t) \le -\omega W_\delta.$$

Solving this inequality, we obtain (2.16). □

4 Discussion

We studied the problem of controlling two low-energy charged particles in a bounded vacuum electromagnetic field by applying electromagnetic forces through the boundary of the domain. Using the perturbed energy method, we designed boundary feedback controllers and proved that the particles under the designed control move to the origin exponentially.

Our results can be extended to the relativistic realm. In that case, the Maxwell equations are unchanged, but the laws of motion must be generalized. Particularly Newton's second law must be revised. In the relativistic version, we must use $\mathbf{f} = m\mathbf{a}$, where $\mathbf{f}$ and $\mathbf{a}$ are the appropriately defined 4-force and 4-acceleration in space-time. In a given coordinate system with an appropriately defined 3-force $\mathbf{F}$, this equation reduces to the Lorentz force law with one minor change: instead of having $\mathbf{F} = m\frac{\partial^2 \mathbf{p}}{\partial t^2}$, we have $\mathbf{F} = \frac{\partial \mathbf{p}}{\partial t}$, where $\mathbf{p}$ is the relativistic momentum.

Our results can be also extended to the motion planning. Let the target orbit $\mathbf{p}_T(t)$ be generated by the system (1.1)-(1.5) and define the energy function

$$\begin{aligned} W(t) &= \frac{1}{2}\int_\Omega \left(\epsilon_0|\mathbf{E}(t,x,y,z) - \mathbf{E}_T(t,x,y,z)|^2 + \frac{1}{\mu_0}|\mathbf{B}(t,x,y,z) - \mathbf{B}_T(t,x,y,z)|^2\right) dV \\ &\quad + \frac{1}{2}\left(k|\mathbf{p}(t) - \mathbf{p}_T(t)|^2 + m|\mathbf{p}'(t) - \mathbf{p}_T'(t)|^2\right), \end{aligned}$$

where $\mathbf{E}_T, \mathbf{B}_T$ are the fields that generate $\mathbf{p}_T(t)$. Using the same mechanism, we can design similar boundary feedback controllers to track $\mathbf{p}_T(t)$.

It would be desirable in practice to design a feedback control using only the outputs $\mathbf{p}$ and $\mathbf{p}'$. A potential way to achieve this is to use the observer-based control design [16, 30]. A possible observer-based output feedback control could be as follows:

$$\begin{aligned}
\epsilon_0\mu_o\frac{\partial \mathbf{E}}{\partial t} &= \nabla\times\mathbf{B}, \quad \text{in } \Omega, && (4.1)\\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}, \quad \text{in } \Omega, && (4.2)\\
\nabla\cdot\mathbf{E} &= \nabla\cdot\mathbf{B} = 0, \quad \text{in } \Omega, && (4.3)\\
\frac{d^2\mathbf{p}}{dt^2} &= \frac{q}{m}\mathbf{E}|_{\mathbf{p}} + \frac{q}{m}\frac{d\mathbf{p}}{dt}\times\mathbf{B}|_{\mathbf{p}}, && (4.4)\\
\mathbf{E}\times\mathbf{n}|_{S_1} &= -c_1\mathbf{B} - \mathbf{p}'(t), \quad \mathbf{B}|_{S_1} = k\mathbf{p}(t) + q\widetilde{\mathbf{E}}|_{\mathbf{p}} + c_3\mathbf{p}'(t), && (4.5)\\
\mathbf{E}\times\mathbf{n}|_{S_2} &= -c_2\mathbf{B}, \quad \mathbf{B}|_{S_2} = \mathbf{p}(t), && (4.6)
\end{aligned}$$

with the observer

$$\begin{aligned}
\epsilon_0\mu_o\frac{\partial \widetilde{\mathbf{E}}}{\partial t} &= \nabla\times\widetilde{\mathbf{B}}, \quad \text{in } \Omega, && (4.7)\\
\frac{\partial \widetilde{\mathbf{B}}}{\partial t} &= -\nabla\times\widetilde{\mathbf{E}}, \quad \text{in } \Omega, && (4.8)\\
\nabla\cdot\widetilde{\mathbf{E}} &= \nabla\cdot\widetilde{\mathbf{B}} = 0, \quad \text{in } \Omega, && (4.9)\\
\frac{d^2\widetilde{\mathbf{p}}}{dt^2} &= -b_1(\mathbf{p}-\widetilde{\mathbf{p}}) - b_2\frac{d}{dt}(\mathbf{p}-\widetilde{\mathbf{p}}), && (4.10)\\
\widetilde{\mathbf{E}}\times\mathbf{n}|_{S_1} &= -c_1\widetilde{\mathbf{B}} - \mathbf{p}'(t), \quad \widetilde{\mathbf{B}}|_{S_1} = k\mathbf{p}(t) + q\widetilde{\mathbf{E}}|_{\mathbf{p}} + c_3\mathbf{p}'(t), && (4.11)\\
\widetilde{\mathbf{E}}\times\mathbf{n}|_{S_2} &= -c_2\widetilde{\mathbf{B}}, \quad \widetilde{\mathbf{B}}|_{S_2} = \mathbf{p}(t). && (4.12)
\end{aligned}$$

To show that this output controller works, we need to estimate the error $\mathbf{E}|_{\mathbf{p}} - \widetilde{\mathbf{E}}|_{\mathbf{p}}$. This is a difficult problem and could not be solved yet.

The existence and uniqueness of a solution of the system (1.1)-(1.5) is an interesting mathematical problem and could not be solved yet. When we tried to use the Galerkin's method, we encountered the difficulty in estimating the solution $\mathbf{E}|_{\mathbf{p}}$ and $\mathbf{B}|_{\mathbf{p}}$. When we tried to use the nonlinear semigroup theory, we had the problem to show the dissipativity of the system.

The convex assumption on the domain Ω could meet practical requirements. But it is an interesting mathematical problem to relax the assumption. For the linear wave equation, this was achieved by using microlocal analysis [19]. Since the system (1.1)-(1.5) is nonlinear and the microlocal analysis is for linear partial differential equation, it will be difficult to attack the problem.

An interesting extension of the problem considered here is to control a flow of continuous electrical charges in an electromagnetic field. The control objective in this case is to drive the flow of the charges to an extreme small region for nuclear fusion. If we denote the density of the charge by ρ, then it should satisfy the convection-diffusion equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \nabla \cdot (\kappa \nabla \rho),$$

where κ is the diffusion coefficient and $\mathbf{v}$ is the velocity field of the charges. In general, the velocity should satisfy the compressible Navier-Stokes equation coupled with the Maxwell's equations. Also heat can be applied to promote the fusion. In this case the diffusion κ is affected by the temperature. This control problem can be reduced to an exact controllability problem: for a desired density profile ρ_d, which is equal to a constant ρ_0 near the origin and zero away from the origin, find a boundary electromagnetic control force such that $\rho(T) = \rho_d$ for some time $T > 0$. This will be a difficult problem.

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