ACTM Regional Calculus Competition
March 2, 2013

1. $\frac{d}{dx}\left(\frac{5}{x}-3\sqrt{x}+4\right)=$

   A. $5\ln(x)-\frac{3}{2}\sqrt{x}+4x$

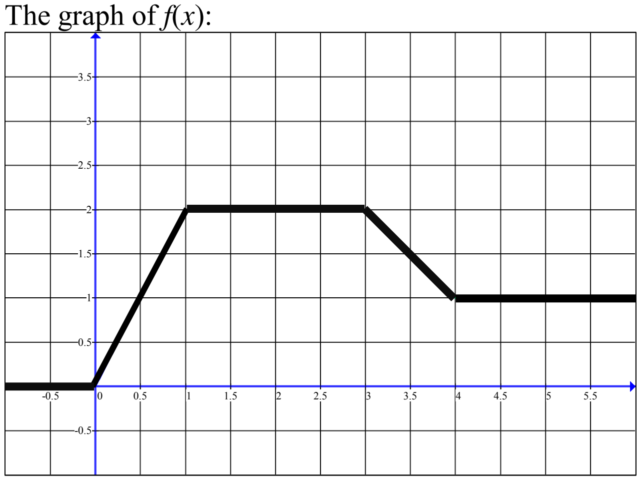
   B. $-\frac{5}{x^2}-\frac{3}{2}\sqrt{x}$

   C. $-\frac{5}{x^2}+\frac{3}{2\sqrt{x}}$

   D. $5x-6\sqrt{x}$

   E. Each of the other answers is incorrect.

2. The graph of $f(x)$:



   $$\int_0^5 f(x)dx=$$

   A. 6.5
   B. 7
   C. 7.5
   D. 8
   E. Each of the other answers is incorrect.

3. Consider the following statements about a function f and its first and second derivatives.
   - I. If $f'(a)=0$ then f must have a local extremum at $x=a$.
   - II. If f has an inflection point at $x=a$ then $f''(a)=0$ or $f''(a)$ is undefined.
   - III. If $f(a)>0$ then f is increasing at $x=a$.
   - IV. If $f''(a)<0$ then f is concave down at $x=a$.

   A. Statements I, II, III, and IV are all true.
   B. Only statements I, II, and IV are true.
   C. Only statements II and IV are true.
   D. Only statements I and IV are true.
   E. Only statements I and III are true.

4. The price p of a stock has been decreasing but is getting ready to bottom out. Let t = time.
   A. $p'(x) < 0$ and $p''(x) < 0$
   B. $p'(x) < 0$ and $p''(x) > 0$
   C. $p'(x) > 0$ and $p''(x) < 0$
   D. $p'(x) > 0$ and $p''(x) > 0$
   E. Each of the other answers is incorrect.

5. Find the equation of the line tangent to $f(x) = 5x^3 + 3x^2 - 40$ at $x = 2$.
   A. $y = 72x - 132$
   B. $y = 36x - 12$
   C. $y = 72x - 12$
   D. $y = 12x - 132$
   E. Each of the other answers is incorrect.

6. The absolute maximum of $f(x) = \cos(2x) - 2\cos(x)$ on $[0, 2\pi]$ occurs at $x =$
   A. $\frac{\pi}{3}$
   B. $\frac{\pi}{2}$
   C. π
   D. $\frac{5\pi}{3}$
   E. $\frac{3\pi}{4}$

7. A region is bounded by the curves $x = 2$, $x = 5$, $y = x^2$, and $y = e^x$. Compute the area of the region. Round your answer to two decimal places.
   A. 96.69
   B. 120.02
   C. 135.02
   D. 102.02
   E. Each of the other answers is incorrect.

8. $\displaystyle\int \frac{\frac{\arctan(x)}{x} - \frac{\ln(x)}{1+x^2}}{(\arctan(x))^2} dx =$
   A. $\frac{1}{x}\ln|\arctan(x)| + C$
   B. $\frac{\ln(x)}{\arctan(x)} + C$
   C. $\frac{\arctan(x)}{x^2+1} + C$
   D. $\ln|\arctan(x)| - \arctan(\ln(x)) + C$
   E. Each of the other answers is incorrect.

9. $\displaystyle\int \frac{\sin\left(\frac{1}{x^2}\right)}{x^3}dx =$

A. $\dfrac{-4\cos\left(\frac{1}{x^2}\right)}{x^4}+C$

B. $\dfrac{1}{2}\cos\left(\dfrac{1}{x^2}\right)+C$

C. $\dfrac{x^3\cos\left(\frac{1}{x^2}\right)\left(\frac{-2}{x^3}\right)-3x^2\sin\left(\frac{1}{x^2}\right)}{x^6}+C$

D. $\dfrac{-4\cos\left(\frac{-1}{x}\right)}{x^4}+C$

E. Each of the other answers is incorrect.

10. A cubic polynomial function $f(x)=ax^3+bx^2+cx+d$ $(a\neq 0)$ has a point of inflection at

A. $x=-\dfrac{b}{2a}$

B. $x=-\dfrac{3a}{2b}$

C. $x=-\dfrac{2b}{3a}$

D. $x=-\dfrac{b}{3a}$

E. Each of the other answers is incorrect.

11. If $b<0$, then $\lim_{x\to\infty}\left(be^{-bx}\right)=$

A. $-\infty$

B. 0

C. 1

D. ∞

E. Each of the other answers is incorrect.

12. Applying L'Hopital's rule, $\lim_{x\to 0^+}\left(\sin(x)\ln(x)\right) =$

A. $\lim_{x\to 0^+}\left(\dfrac{\cos(x)}{x}\right)$

B. $\lim_{x\to 0^+}\left(\dfrac{\frac{1}{x}}{\cos(x)}\right)$

C. $\lim_{x\to 0^+}\left(\dfrac{\frac{1}{x}}{-\csc(x)\cot(x)}\right)$

D. $\lim_{x\to 0^+}\left(\dfrac{\cos(x)}{\frac{1}{x}}\right)$

E. Each of the other answers is incorrect.

13. $\dfrac{d}{dx}\int_2^x \ln(t)\,dt =$

A. $\ln(x)$

B. $x\ln(x) - x - 2\ln(2) + 2$

C. $\ln(x) - \ln(2)$

D. $\dfrac{1}{x} - \dfrac{1}{2}$

E. Each of the other answers is incorrect.

14. $\lim_{x\to 2}\left(\dfrac{2x^2 - 3x - 2}{x-2}\right) = 5$. By the definition of a limit, there is a positive real number δ such that $\left|\dfrac{2x^2 - 3x - 2}{x-2} - 5\right| < 0.1$ if $0 < |x-2| < \delta$. The largest valid value of δ is

A. 0.02
B. 0.05
C. 0.1
D. 0.2
E. 0.5

15. $\lim_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x} =$

A. $\lim_{x\to 0}\left(\dfrac{2+x-2}{x\left(\sqrt{2+x}+\sqrt{2}\right)}\right)$

B. $\lim_{x\to 0}\left(\dfrac{2+x-2}{x\left(\sqrt{2+x}-2\right)}\right)$

C. $\lim_{x\to 0}\left(\dfrac{\sqrt{2}+\sqrt{x}-\sqrt{2}}{x}\right)$

D. $\lim_{x\to 0}\left(\dfrac{\sqrt{2+x}+\sqrt{2}}{x\left(\sqrt{2+x}+2\right)}\right)$

E. Each of the other answers is incorrect.

16. Approximate $\frac{d}{dx}\left(x^{3x}\right)\Big|_{x=2}$ to two decimal places.

A. 192.00
B. 133.08
C. 256.25
D. 325.08
E. 64.00

17. $f(x)=\sin\left(x^2\right)$. The second derivative $f''(x) =$

A. $-\sin\left(x^2\right)$

B. $-2\sin\left(x^2\right)$

C. $2\cos\left(x^2\right)-2x\sin\left(x^2\right)$

D. $2\cos\left(x^2\right)-4x^2\sin\left(x^2\right)$

E. Each of the other answers is incorrect.

18. Approximate $\int_1^9 x^2 dx$ using the Trapezoidal Rule with 4 divisions.

A. 124
B. 168
C. 243
D. 248
E. 328

19. A freezer depreciates at a rate of 18% per year. It was purchased new for $755. How fast is it depreciating when it is exactly 5 years old?

A. $34.50 per year
B. $41.50 per year
C. $55.55 per year
D. $67.83 per year
E. Each of the other answers is incorrect.

20. A company is marketing bottles of oil. The amount they are willing to supply to the market is dependent on the price they can get per bottle. They have determined that the following function closely approximates how many bottles they are willing to supply as a function of the price per bottle: $S(p) = 8620\left(e^{0.27p} - 2\right)$. Compute and interpret S'(3.5).
   A. S ' (3.5) ≈ 4938. When the price is \$3.50 per bottle the company is willing to supply 4938 bottles.
   B. S ' (3.5) ≈ 4938. When the price is \$49.38 per bottle the company is willing to supply 3.5 bottles.
   C. S ' (3.5) ≈ 5988. When the price is \$3.50 per bottle the company is willing to supply 5988 bottles.
   D. S ' (3.5) ≈ 5988. When the price is \$3.50 per bottle the company is willing to supply 5988 additional bottles for every \$1 increase in the price per bottle.
   E. Each of the other answers is incorrect.

21. The flow of income in dollars per day from vending machines belonging to a certain company is given by the formula $f(t) = 235e^{0.03t}$ where *t* is the number of days since the beginning of 2013. Compute the total amount of income from this source for January 2013.
   A. \$12,020.32
   B. \$595.61
   C. \$17.87
   D. \$360.61
   E. Each of the other answers is incorrect.

22. Consider the following table of values from a differentiable function:

x	2	4	6	8
***f*(*x*)**	2.5	3.7	9.8	15.7

   What is the *best* estimate of *f* '(6) given this information?
   A. 2.95
   B. 3
   C. 3.05
   D. 5.9
   E. 9.8

23. Let *R* be the region between the graphs of $y = 1$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$. The volume of the solid obtained by revolving *R* about the <u>*x*-axis</u> is given by
   A. $2\pi\int_0^{\frac{\pi}{2}} x \sin x dx$
   B. $2\pi\int_0^{\frac{\pi}{2}} x \cos x dx$
   C. $\pi\int_0^{\frac{\pi}{2}} (1 - \sin x)^2 dx$
   D. $\pi\int_0^{\frac{\pi}{2}} \sin^2 x dx$
   E. $\pi\int_0^{\frac{\pi}{2}} (1 - \sin^2 x) dx$

Name ________________________________

Tie Breaker 1:

Consider the finite region bounded by curves $y = 3^x$ and $y = -2x^2 + 8x + 1$. Sketch a graph of the region and compute its area exactly and rounded to four decimal places.

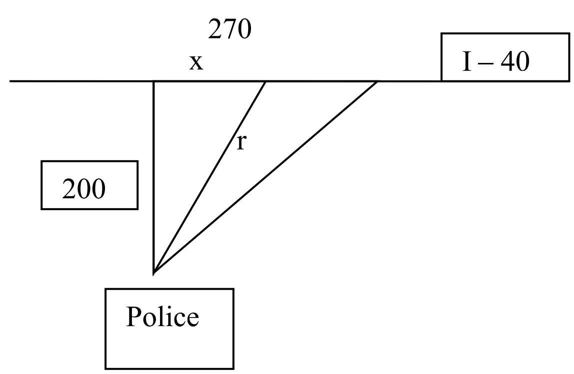
Tie Breaker 2:

The volume of a typical soda can is 355 ml. It is made in the shape of a cylinder. The top of the can is approximately three times as thick as the sides. Assume that the cost of the can is proportional to the amount of aluminum used. What dimensions should the can have in order to minimize the total cost of making the can?

Name ________________________________

Tie Breaker 3:

A highway patrol officer's radar unit is parked behind a billboard 200 feet from a long straight stretch of I-40. Down the highway, 270 feet from the point on the highway closest to the officer is an emergency call box. The officer points the radar gun at the call box and measures the speed of the traffic passing the box. Actually the radar unit measures how fast the distance between the radar unit and the traffic is increasing at the time the automobile passes the call box. If the posted speed limit is 75 miles per hour what is the maximum speed measured by the call box which would correspond to a legal speed?



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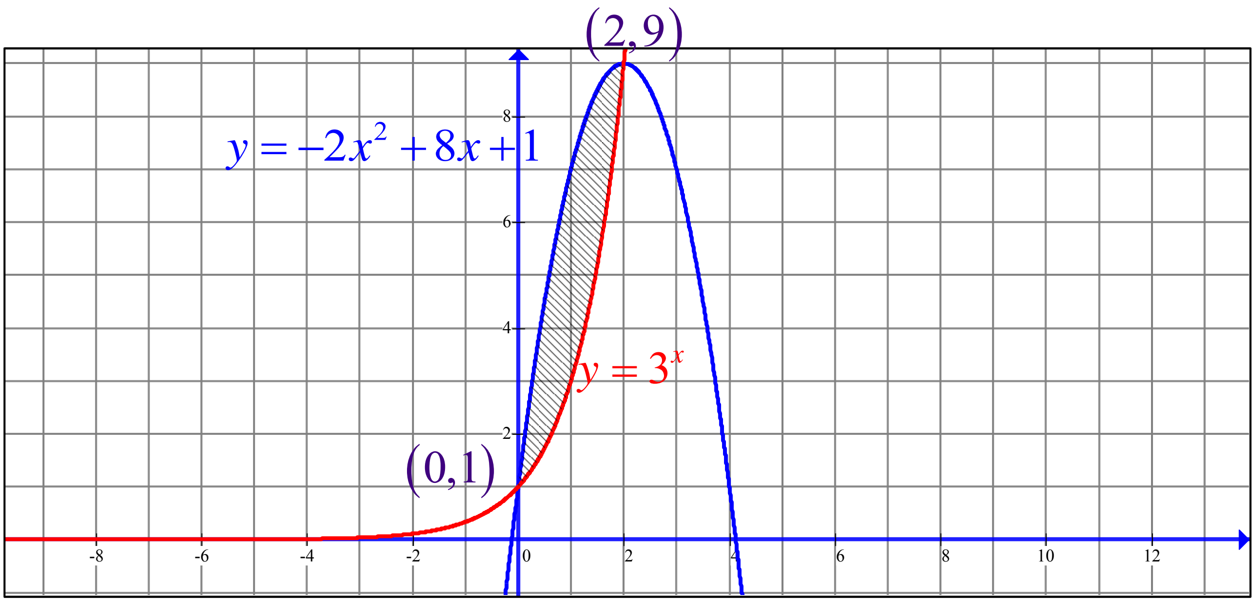
Key

1.	**E**	$\left(-\frac{5}{x^2}-\frac{3}{2\sqrt{x}}\right)$
2.	**C**	
3.	**C**	
4.	**B**	
5.	**A**	
6.	**C**	
7.	**D**	
8.	**B**	
9.	**B**	
10.	**D**	
11.	**A**	
12.	**C**	
13.	**A**	
14.	**B**	
15.	**A**	
16.	**D**	
17.	**D**	
18.	**D**	
19.	**C**	
20.	**D**	
21.	**A**	
22.	**B**	
23.	**E**	

Tie Breaker 1:

Consider the finite region bounded by curves $y = 3^x$ and $y = -2x^2 + 8x + 1$. Sketch a graph of the region and compute its area exactly and rounded to four decimal places.



$$A = \int_0^2 \left(\left(-2x^2 + 8x + 1\right) - 3^x \right) dx$$

$$= \left(-\tfrac{2}{3}x^3 + 4x^2 + x - \tfrac{1}{\ln(3)} 3^x \right) \Big]_{x=0}^{x=2}$$

$$= \left(-\tfrac{2}{3}(2)^3 + 4(2)^2 + (2) - \tfrac{1}{\ln(3)} 3^{(2)} \right) - \left(-\tfrac{2}{3}(0)^3 + 4(0)^2 + (0) - \tfrac{1}{\ln(3)} 3^{(0)} \right)$$

$$= -\tfrac{16}{3} + 16 + 2 - \tfrac{9}{\ln(3)} + \tfrac{1}{\ln(3)}$$

$$= -\tfrac{16}{3} + 18 - \tfrac{8}{\ln(3)}$$

$$= -\tfrac{16}{3} + \tfrac{54}{3} - \tfrac{8}{\ln(3)}$$

$$= \boxed{\tfrac{38}{3} - \tfrac{8}{\ln(3)}}$$

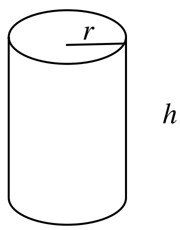
$$\approx 5.3848$$

Name ______________________________

Tie Breaker 2:

The volume of a typical soda can is 355 ml. It is made in the shape of a cylinder. The top of the can is approximately three times as thick as the sides. Assume that the cost of the can is proportional to the amount of aluminum used. What dimensions should the can have in order to minimize the total cost of making the can?

Let V = volume, r = radius, h = height, C = cost,
k = cost per square centimeter of sides.



$$V = Bh = \pi r^2 h$$
$$355ml = \pi r^2 h$$
$$355cm^3 = \pi r^2 h$$
$$h = \frac{355cm^3}{\pi r^2}$$

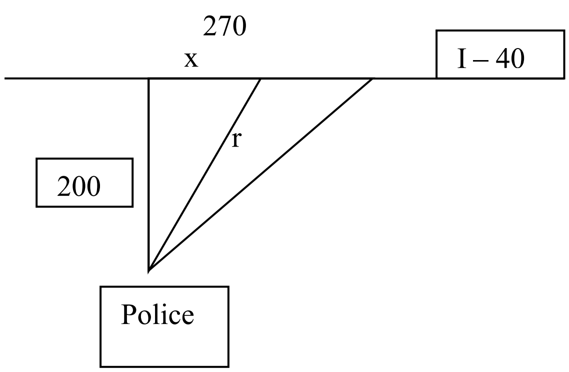
$$C = k\left[4\pi r^2 + 2\pi r h\right]$$
$$C = k\left[4\pi r^2 + 2\pi r \frac{355cm^3}{\pi r^2}\right]$$
$$C(r) = 2k\left[2\pi r^2 + 355r^{-1}\right]$$
$$C'(r) = 2k\left[4\pi r - 355r^{-2}\right]$$
$$0 = C'(r)$$
$$0 = 2k\left[4\pi r - 355r^{-2}\right]$$
$$0 = 4\pi r - 355r^{-2}$$
$$355r^{-2} = 4\pi r$$
$$r^3 = \frac{355}{4\pi}$$
$$r = \sqrt[3]{\frac{355}{4\pi}} \doteq 3.04559976066cm$$
$$h = \frac{355cm^3}{\pi\left(\sqrt[3]{\frac{355}{4\pi}}\right)^2} \doteq 12.1823990427cm$$

Under these conditions the cost is minimized when the radius of the can is 3.05 cm and its height is 12.18 cm. (Note that in spite of our simplifications this is quite close to the actual dimensions of a soda can.)

Name ______________________________

Tie Breaker 3:

A highway patrol officer's radar unit is parked behind a billboard 200 feet from a long straight stretch of I-40. Down the highway, 270 feet from the point on the highway closest to the officer is an emergency call box. The officer points the radar gun at the call box and measures the speed of the traffic passing the box. Actually the radar unit measures how fast the distance between the radar unit and the traffic is increasing at the time the automobile passes the call box. If the posted speed limit is 75 miles per hour what is the maximum speed measured by the call box which would correspond to a legal speed of the traffic?



Let A be the point on the highway closest to the police officer. Let x be the distance the car is from point A. Let r be the distance from the police officer to the car. Notice that both r and x are functions of time. Notice that we have a set of right triangles with one leg fixed and the lengths of the other two sides varying. We have the following Pythagorean relationship:

$$r^2 = x^2 + 200^2$$

Differentiating both sides with respect to time we get a related rate differential equation.

$$\frac{d}{dt}\left(r^2\right) = \frac{d}{dt}\left(x^2 + 200^2\right)$$
$$2r\frac{dr}{dt} = 2x\frac{dx}{dt}$$
$$\frac{dr}{dt} = \frac{x}{r}\frac{dx}{dt}$$

At the point the measurement is made the distances are

$$x = 270 \Rightarrow r = \sqrt{270^2 + 200^2} = \sqrt{112900} = 10\sqrt{1129}\,.$$

We know that the maximum legal speed on the highway is 75 mi/hr so we have $\frac{dx}{dt} = 75$. Substituting and solving we get:

$$\begin{aligned}\frac{dr}{dt} &= \frac{x}{r}\frac{dx}{dt} \\ &= \frac{270\,ft}{10\sqrt{1129}\,ft}\left(75\,\tfrac{mi}{hr}\right) \\ &= \frac{2025}{\sqrt{1129}}\,\tfrac{mi}{hr} \\ &\cong 60.2667895\,\tfrac{mi}{hr}\end{aligned}$$

Thus the maximum legal recorded speed is about 60.3 mi/hr.