ACTM State Calculus Competition
Saturday April 30, 2011

Instructions: For questions 1 through 25, mark the best answer choice on the answer sheet provided. Afterward completing questions 1 through 25, attempt the tie-breaker questions in sequential order (do #1 first, followed by #2, and then #3 last). Be sure that your name is printed on each of the tiebreaker questions. Unless otherwise stated, assume all variables are real and all functions are continuous over relevant domains. Assume all angles are in Radians. Good luck!

1. Consider the following limit:
$$\lim_{x\to 3} 2x+1$$
Given $\varepsilon = .01$, determine the maximum value for δ to satisfy the $\epsilon - \delta$ definition of the limit.
    a. $.005$
    b. $.01$
    c. $.02$
    d. 7
    e. Cannot be determined.

2. Determine the slope of the normal line to the function $\psi(\theta) = \sin\left(\frac{1}{\theta}\right)$ at the point $\theta = \pi$.
    a. $-\cos(1)$
    b. $-\frac{\cos\left(\frac{1}{\pi}\right)}{\pi^2}$
    c. $-\frac{1}{\sin\left(\frac{1}{\pi}\right)}$
    d. $\sin\left(\frac{1}{\pi}\right)$
    e. $\frac{\pi^2}{\cos\left(\frac{1}{\pi}\right)}$

3. Apple Inc.'s stock price has shown exponential growth since 1980. The stock price can be modeled with the equation $A(t) = 3e^{.154t}$, where t is in years since 1980. What does this model predict the growth rate per year is this year (2011)?
    a. \$3.50 per year
    b. \$54.70 per year
    c. \$108.48 per year
    d. \$355.18 per year
    e. Cannot be determined

4. Find the oblique asymptote, if one exists, to the following function.
$$y = \frac{x^2+3x+2}{x-2}$$
    a. $y = \frac{12}{x-2}$
    b. $x = -1$
    c. $y = 2$
    d. $y = x + 5$
    e. No oblique asymptote exists.

5. Determine $f(x)$ by anti-differentiating twice.

$$f''(x) = 4x + 3$$

   a. $f(x) = 0$
   b. $f(x) = \frac{2x^3}{3} + \frac{3x^2}{2}$
   c. $f(x) = \frac{2x^3}{3} + \frac{3x^2}{2} + C_1$
   d. $f(x) = \frac{2x^3}{3} + \frac{3x^2}{2} + C_1x$
   e. $f(x) = \frac{2x^3}{3} + \frac{3x^2}{2} + C_1x + C_2$

6. Determine the total area bounded by the equation $(x) = \frac{|2x|}{x}$, the $x-$ axis, the lines $x = -4$ and $x = 3$.
   a. $Area = -2\ units^2$
   b. $Area = -1\ unit^2$
   c. $Area = 2\ units^2$
   d. $Area = 7\ units^2$
   e. $Area = 14\ units^2$

7. Find $G'(x)$ if $G(x) = \int_0^x \sec^3(t)\tan(t)\,dt$
   a. $G'(x) = \frac{1}{3}\sec(x)^3$
   b. $G'(x) = \frac{\sin(x)}{\cos(t)^4}$
   c. $G'(x) = \frac{\cos(x)}{4\cos(t)^3\sin(t)}$
   d. $G'(x) = \sec^3(x)\tan(x)$
   e. Unable to evaluate.

8. Evaluate the following indefinite integral:

$$\int \sin^3(2x)\cos(2x)\,dx$$

   a. $\frac{\sin^4(x^2)}{4} + C$
   b. $\frac{\sin^4(2x)}{8} + C$
   c. $-\frac{\cos^4(2x)}{4} + C$
   d. $-\frac{\cos^4(2x)}{8} + C$
   e. Unable to integrate

9. Determine the maximum number of inflection points that a quadratic function of the form $f(x) = ax^2 + bx + c$ may have.
   a. 1
   b. 2
   c. $-\frac{b}{2a}$
   d. $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
   e. It has no inflection points.

10. Find the average depth of a 2–foot tall water trough with a cross sectional area bounded by the equation $T(x) = \frac{8}{9}x^2 - \frac{8}{3}x + 2$ and $W(x) = 2,\ 0 \le x \le 3,$ where x is in feet.
   a. $\frac{2}{3}\ feet$
   b. $1\ foot$
   c. $\frac{4}{3} feet$
   d. $3\ feet$
   e. $6\ feet$

11. The radius of a circular oil spill is growing at a constant rate of 2 km per day. At what rate per day is the area of the spill growing 4 days after it began? Assume the thickness is constant.
   a. $4\pi \frac{km^2}{day}$
   b. $16\pi \frac{km^2}{day}$
   c. $32\pi \frac{km^2}{day}$
   d. $64\pi \frac{km^2}{day}$
   e. Unable to determine from information given.

12. Find the maximum volume of a rectangular open-top box made from a piece of cardboard measuring 24 inches long and 9 inches wide by cutting out identical squares from the four corners and turning up the sides.
   a. $2\ in^3$
   b. $4\ in^3$
   c. $200\ in^3$
   d. $216\ in^3$
   e. $400\ in^3$

13. What is/are the region(s) where the following function is concave up?
$$f(x) = \frac{x}{x^2 + 4}$$
   a. $(-2,2)$
   b. $(-\infty,-2) \cup (2,\infty)$
   c. $\left(-2\sqrt{3},0\right) \cup \left(2\sqrt{3},\infty\right)$
   d. $\left(-\infty,2\sqrt{3}\right) \cup \left(0,-2\sqrt{3}\right)$
   e. The function has no regions where it is concave up.

14. Consider the function $f(x) = \frac{1}{3}x^3 - 4x - 6$ with the domain $[-3, 1]$. Determine the coordinate of the minimum.
    a. $(2, 11.3)$
    b. $(-3, -3)$
    c. $(-2, -0.7)$
    d. $(1, -9.67)$
    e. $(-10, -299.3)$

15. One of the most famous symbols in all of science fiction is the double parabola shape found in the *Star Trek* series. The equations are $T(x) = -.7x^2 + .1x + 9$ and $B(x) = -.2x^2 + .1x - 3.7$. Find the area bounded between these two parabolas.
    a. $8.92\ units^2$
    b. $12.41\ units^2$
    c. $43.05\ units^2$
    d. $54.36\ units^2$
    e. $85.34\ units^2$

16. Calculate the volume of the solid of revolution generated by the region bounded in the first quadrant by the curve $\mathrm{y} = 3\mathrm{x} - \mathrm{x}^3$, the y–axis, and the line $\mathrm{y} = 2$, rotated around the y–axis. Use the shell method.
    a. $\mathrm{V} = \frac{2}{5}\pi$ units^3
    b. $\mathrm{V} = \frac{5}{4}\pi$ units^3
    c. $\mathrm{V} = \frac{8}{5}\pi$ units^3
    d. $\mathrm{V} = \frac{16}{5}\pi$ units^3
    e. $\mathrm{V} = \frac{68}{35}\pi$ units^3

17. Determine the sum of the following:

$$\sum_{i=0}^{20}(3i + 1)(i^2 + 2)$$

    a. 136,470
    b. 136,472
    c. 1,891,500
    d. 1,895,712
    e. The summation has no value

18. The concentration of a certain drug in the bloodstream can be approximated by the equation $C(t) = 1.4(6t + 3)^{-1/2}$, where $C(t)$ is the concentration (in percent) and t is in hours after taking the medicine. Find the rate of change of concentration per hour at $t = 5$ hours
    a. 2.2% decrease per hour
    b. 3.7% decrease per hour
    c. 4.8% decrease per hour
    d. 24.4% increase per hour
    e. 28.6% decrease per hour

19. The hyperbolic functions are analogous to the standard trig functions, but they are defined in terms of a hyperbola instead of a circle. One of these functions is *hyperbolic tangent*, which is defined as $tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Calculate the first derivative of the $tanh(x)$ function.
    a. $tanh'(x) = \frac{2}{e^{2x}+1}$
    b. $tanh'(x) = \frac{e^x+e^{-x}}{e^x-e^{-x}}$
    c. $tanh'(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$
    d. $tanh'(x) = \frac{2}{e^x+e^{-x}+1}$
    e. $tanh'(x) = \frac{4}{e^{2x}+e^{-2x}+2}$

20. Determine the domain of the following function:

$$G(x) = \frac{x^2 - 4}{x^4 - 16}$$

    a. $\{x | x \in \mathbb{R}\}$
    b. $\{x | x \neq 0\}$
    c. $\{x | x \neq 2, x \neq -2\}$
    d. $\{x | x \neq 4, x \neq -4\}$
    e. $\{x | x \neq -4, x \neq -2, x \neq 2, x \neq 4\}$

21. L'Hôpital's rule can be applied to solve limits involving indeterminant forms. Which of the following is <u>not</u> an indeterminant form?
    a. 0^0
    b. $0 - 0$
    c. $\infty - \infty$
    d. $0 \cdot \infty$
    e. ∞^0

22. One day while Cookie Monster was sitting in the top of a 25-meter tree, he dropped a cookie. Assuming that his cookie falls only under the influence of gravity, what is the acceleration at the instant it hits the ground?
    a. $0\ m/sec^2$
    b. $2.3\ m/sec^2$
    c. $9.8\ m/sec^2$
    d. $22.1\ m/sec^2$
    e. $25\ m/sec^2$

23. Calculate the following anti-derivative using a trig substitution.

$$\int \frac{1}{4+\theta^2}\,d\theta$$

a. $\frac{1}{4}\ln\left(\frac{\theta+2}{\theta-2}\right)+C$
b. $\sin^{-1}\left(\frac{\theta}{2}\right)+C$
c. $\frac{1}{4}\sqrt{4-\theta^2}+C$
d. $\frac{1}{2}\tan^{-1}\left(\frac{\theta}{2}\right)+C$
e. $\frac{1}{2}\ln(\theta^2+4)+C$

24. Consider the function,

$$g(r)=\begin{cases}2r+3 & if\ r<3\\ 2 & if\ r=3\\ 7-2r & if\ r>3\end{cases}$$

Evaluate the following limit

$$\lim_{x\to 3^+} g(r)$$

a. 1
b. 2
c. 3
d. 9
e. The limit does not exist.

25. The Mean Value Theorem states that for any function $f(x)$ that is continuous on $[a, b]$ and differentiable on (a, b), there exists some c in the interval (a, b) such that the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is parallel to the tangent line to f at $(c, f(c))$. The following function satisfies the Mean Value Theorem on the given interval. Find a suitable value for c that satisfies the conclusion of the Mean Value Theorem.

$$f(x)=x^2+2x-1 \quad \text{on the interval } [0,3]$$

a. $c=1.5$
b. $c=5$
c. $c=15$
d. $c=\sqrt{2}-1$
e. $c=\sqrt{5}-1$

ACTM State Calculus Competition
Tie Breaker Questions
April 30, 2011

Name ____________________________________

School/Teacher __________________________

Reminders: Attempt these tiebreaker questions after you have finished all the multiple-choice questions. Attempt the tiebreaker questions in sequential order (Do #1 first, followed by #2, and then #3 last).

1. Given the function $f(x) = \begin{cases} 2x - a & if\ x < -3 \\ ax + 2b & if - 3 \le x \le 3 \\ b - 5x & if\ 3 < x \end{cases}$, find the values of a and b such that $\lim_{x \to -3} f(x)$ and $\lim_{x \to 3} f(x)$ both exist.

2. Given the formula for a general cubic equation, $f(x) = ax^3 + bx^2 + cx + d$.
   a. Derive a formula to find the x-value(s) of the relative extrema.

   b. What are the conditions required to guarantee that there are two distinct extrema?

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3. Find the equation(s) of the tangent line(s) to the following equation. Use Implicit Derivatives. Assume $y = y(x)$.

$$y^2 - xy + 3 = 0 \quad \text{at } x = -4$$

ACTM State Calculus Competition
SOLUTIONS
April 30, 2011

Multiple Choice Answers

1......A

2......E

3......B

4......D

5......E

6......E

7......D

8......B

9......E

10....C

11....C

12....C

13....C

14. ..D

15. ..E

16. ..A

17. ..B

18. ..A

19. ..E

20. ..C

21. ..B

22. ..C

23. ..D

24. ..A

25. ..A

Tie Breaker Question 1 Solution

1. Given the function $f(x) = \begin{cases} 2x - a & if\ x < -3 \\ ax + 2b & if -3 \leq x \leq 3 \\ b - 5x & if\ 3 < x \end{cases}$, find the values of a and b such that $\lim_{x \to -3} f(x)$ and $\lim_{x \to 3} f(x)$ both exist.

This is a system of equations problem in disguise. Since $\lim_{x \to -3} f(x)$ and $\lim_{x \to 3} f(x)$ both exist, the limits to the left and right must be equal. Thus, check the following equations:

$$\begin{aligned} \lim_{x \to -3^-} f(x) &= \lim_{x \to -3^+} f(x) \\ 2(-3) - a &= -3a + 2b \\ -6 - a &= -3a + 2b \\ -6 &= -2a + 2b \end{aligned}$$

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^+} f(x) \\ 3a + 2b &= b - 5(3) \\ 3a + 2b &= b - 15 \\ 3a + b &= -15 \\ b &= -3a - 15 \end{aligned}$$

Substitute from the right column,

$$\begin{aligned} -6 &= -2a + 2(-3a - 15) \\ -6 &= -2a - 6a - 30 \\ -6 &= -8a - 30 \\ 24 &= -8a \\ -3 &= a \end{aligned}$$

Substitute from the left column,

$$\begin{aligned} b &= -3(-3) - 15 \\ b &= -6 \end{aligned}$$

The final function is $f(x) = \begin{cases} 2x + 3 & if\ x < -3 \\ -3x - 12 & if -3 \leq x \leq 3, \\ -6 - 5x & if\ 3 < x \end{cases}$

Tie Breaker Question 2 Solution

2. Given the formula for a general cubic equation, $f(x) = ax^3 + bx^2 + cx + d$.
   a. Derive a formula to find the relative extrema.
      i. Find the derivative of the formula and set it equal to 0.

      $$f'(x) = 3ax^2 + 2bx + c = 0$$

      use the quadratic formula, with $A = 3a, B = 2b$, and $C = c$.

      $$x = \frac{-(2b) \pm \sqrt{(2b)^2 - 4(3a)(c)}}{2(3a)}$$

      $$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

      This formula will give the $x-$ values for the extrema.

   b. What are the conditions required to guarantee that there are two distinct extrema?
      i. The discriminate must be positive for the quadratic equation to have two distinct solutions. Thus, the condition is that

      $$4b^2 - 12ac > 0 \text{ or } b^2 - 3ac > 0$$

Tie Breaker Question 3 Solution

3. Find the equation(s) of the tangent line(s) to the following equation. Use Implicit Derivatives. Assume that $y = y(x)$.

$$y^2 - xy + 3 = 0 \quad \text{at } x = -4$$

i. First, find the y–coordinate by substituting x and solving.

$$y^2 - (-4)y + 3 = 0$$
$$y^2 + 4y + 3 = 0$$
$$(y+1)(y+3) = 0$$
$$y = -1, \qquad y = -3$$

Thus there are two coordinates that satisfy this equation,

$$P_1 = (-4, -1) \qquad\qquad P_2 = (-4, -3)$$

ii. Second, find the implicit derivative.

$$2y \cdot y' - xy' - y = 0$$

Solve for y',

$$y' = \frac{-y}{x - 2y}$$

iii. Use the implicit derivative to find the slopes,

$$y' = \frac{-(-1)}{-4 - 2(-1)} = -\frac{1}{2} \qquad\qquad y' = \frac{-(-3)}{-4 - 2(-3)} = \frac{3}{2}$$
$$m_1 = -\frac{1}{2} \qquad\qquad m_2 = \frac{3}{2}$$

iv. Now given these points and slopes, find the y-intercepts,

$$-1 = -\frac{1}{2}(-4) + b \qquad\qquad -3 = \frac{3}{2}(-4) + b$$
$$b_1 = -3 \qquad\qquad b_2 = 3$$

v. Now pull everything together into the final equations,

$$\boldsymbol{y = -\frac{1}{2}x - 3} \qquad\qquad \boldsymbol{y = \frac{3}{2}x + 3}$$

These are the two tangent lines at $x = -4$.