ACTM State Math Contest
Calculus
March 6, 2010

1. Find h(2) such that the function h(x) = $\frac{x^2 + 3x - 10}{x - 2}$ is continuous at x = 2 .

   a. 0 b. 2 c. 7 d. Not possible e. None of the above

2. Evaluate $\lim_{x \to 0} \frac{a^x - b^x}{x}$

   a. 0 b. 1 c. ln (a/b) d. does not exist e. none of the above

3. The function f(x) = $\begin{cases} \frac{7|x| + 5x}{7|x| - 5x} & if \quad x \neq 0 \\ 6 & if \quad x = 0 \end{cases}$ is

   a. continuous at x = 0
   b. right continuous at x = 0
   c. left continuous at x = 0
   d. does not exists
   e. none of the above

4. $\lim_{x \to \pi/4} \frac{\sin^2 x - 1/2}{x - \pi/4}$ is equal to

   a. -1 b. 0 c. 1 d. does not exist e. none of the above

5. Evaluate $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}$

   a. 0 b. 1 c. 8 d. does not exist e. none of the above

6. Let f(x) = | 7 - 6x - x^2 |. Which one of the following statements is true?

a. f (x) is nowhere differentiable on the number line
b. f(x) is differentiable at every point on the number line
c. f(x) is differentiable at every point of the number line except for x =1
d. f(x) is differentiable at every point of the number line except for x = -7
e. none of the above

7. The function y = $\frac{\ln x}{x}$ is increasing on the interval

a. (0, ∞) b. (1, ∞) c. (e, ∞) d. It is never increasing e. none of the above

8. If y = $\sin^2(\cos 2x)$, find $\frac{dy}{dx}$ when x = $\pi/8$.

a. $\sqrt{2}\cos(\sqrt{2})$ b. - $\sqrt{2}\sin(\sqrt{2})$ c. $\sqrt{2}\sin(\sqrt{2})$
d. - $\sqrt{2}\cos(\sqrt{2})$ e. none of the above

9. Find the equation of the tangent line to the graph of $x^2y + xy^2 = 6$ at (2,1).

a. 5x – 8y = 18 b. 5x -8y = -18 c. 5x + 8y = -18 d. 5x + 8y = 18

e. none of the above

10. Let y be a function of x which satisfies ln (x+y) – $e^{2xy} + 1 = 0$. Find $y'(0)$.

a. -1 b. -1/2 c. 0 d. 1 e. none of the above

11. Find equation of the tangent line to the circle $x^2 + y^2 + 2x + 6y = 0$ at the point (-2,0).

a. x – 3y - 2 = 0
b. x+3y + 2 = 0
c. x+3y – 2 = 0
d. x – 3y + 2 = 0
e. None of the above

12. Find the limit of the Riemann sum $\frac{1}{n}\left[\frac{1}{n^2}+\frac{4}{n^2}+\ldots+\frac{(n-1)^2}{n^2}+1\right]$ as n tends to ∞.

a. 1/3 b. 2/3 c. 1 d. does not exist e. None of the above

13. Find the 31 st derivative of the function f(x) = Sin 2x

a. -2^{31} cos 2x b. 2^{31} cos2x c. 2^{31} sin2x d. -2^{31} sin2x
e. None of the above

14. The voltage V (volts), current I (amperes) and resistance R (Ohms) of an electric circuit is related by the equation V = I R. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote the time in seconds. Find the rate at which R is changing when V = 12 volts and I = 2 amp.

a. -3/2 b. --1 c. 3/2 d. 5/3 e. none of the above

15. The tangent to the function y = f(x) forms an angle $\pi/6$ at x =1 and an angle $\pi/4$ at x =3 with the X-axis. Then $\int_1^3 f'(x)\, f''(x)\, dx$ equals

a. 1/3 b. $\frac{3+\sqrt{3}}{3}$ c. $\frac{3-\sqrt{3}}{3}$ d. 3 e. none of the above

16. Find f(4) if $\int_0^{x^2} f(t)\, dt = x\cos \pi x$.

a. -1/2 b. 1/4 c. 1/2 d. 1 e. None of the above

17. Evaluate $\int_0^5 |x^2 - 5x + 4|\, dx$.

a. 2/3 b. 9/2 c. 22/6 d. 49/6 e. None of the above.

18. $\int_0^{\pi/2} \sin^6 x \cos^3 x\, dx$ is

a. 0 b. 1/28 c. 1/63 d.2/63 e. None of the above

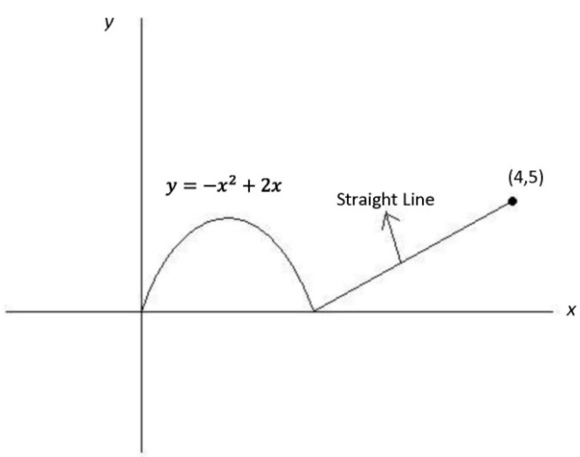
19. Evaluate $\int \frac{\cos 2x}{(\cos x - \sin x)^2}\, dx$

a. - ½ ln |cos x + sin x| + c
b. ½ ln |1 + sin 2x| + c
c. - ½ ln |cos x – sin x| + c
d. - ½ ln|1 – sin 2x | + c
e. None of the above

20. The maximum value of the function $y = x^3 - 3x$ on [0,2] is

a. 0 b. -2 c. 1 d. 2 e. None of the above

21. Find the average value of the function shown in the figure.



a. 19/12 b. 19/6 c. 7/4 d. 5/2 e. None of the above

22. The position function of a particle at a time t (in seconds) is given by $s = 4t^3 - 18t^2 - 15$ measured in feet. Find the velocity (ft/sec) when the acceleration is zero.

a. -27 b. -25 c. 25 d. 27 e. None of the above

23. The area bounded between the curves $y = x^2$ and $y = x^3$ is :

a. 1/6 b. 1/12 c. 1 d. 2 e. None of the above

24. Find all numbers c for the function $f(x) = \ln(x-1)$ on [2,4] that satisfy the conclusion of the Mean Value Theorem.

a. ln (3/2) b. $\frac{3}{\ln(3/2)}$ c. $\frac{2+\ln(3)}{\ln(3)}$ d. 3 e. None of the above

25. Two sides of a triangle are 3 meters and 4meters in length and the angle between them is increasing at a rate of 0.03 radians per second. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$.

a. 0.09 sq meters/sec
b. 0.625 square meters/sec
c. 0.18 square meters/sec
d. 0.15 square meters/sec
e. none of the above

Tie Breaker #1 Name:________________________

For what values of a and b will f(x) = $\begin{cases} ax & if \quad x < 2 \\ ax^2 - bx + 3 & if \quad x \geq 2 \end{cases}$

be differentiable for all values of x?

Solution: Since f has to be continuous at x =2, the left limit of f at x =2 must be equal to the right limit of f at x = 2. This means, 2a = 4a – 2b + 3 or

2a – 2b + 3 = 0 (1)

Since f has to be differentiable at x =2, the left derivative at x =2 must be equal to the right derivative at x = 2. This means, a = 4a – b or

3a - b = 0 (2).

Solving the above system of equations (1) and (2) gives us a = 3/4 and b = 9/4.

Tie Breaker #2 Name:____________________

Find f($\pi/2$) from the following information.

a. f(x) is a positive continuous function
b. The area under the curve y = f(x) from x =0 to x = a is $\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$.

Solution: By (a) and (b), we have

$$\int_0^x f(t)\, dt = \frac{x^2}{2} + \frac{x}{2}\sin x + \frac{\pi}{2}\cos x$$

By differentiating both sides of the above equation, we get

f(x) = x + (sin x)/2 + (x cos x)/2 – ((pi)/2)sin x

Therefore

f($\pi/2$) $= \frac{1}{2}$.

Tie Breaker #3 Name:___________________

Let f(x) = $\begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Show that f is continuous only at x = 0.

Solution: If x is no-zero rational number, by selecting a sequence of irrational number converging to x, we see that f is not continuous at x. On the other hand, if x is an irrational number, by selecting a sequence of rational numbers converging to x, we see that f is not continuous at x. Finally, if a sequence $(x_n\}$ converges to zero, then it is obvious that $f(x_n)$ will also converge to zero. Therefore, f is continuous only at x = 0.

Key – ACTN Regional Math Contest (Calculus Exam)

1. C	9. D	17. D	25 A
2. C	10. D	18. D	
3. B	11. D	19. D	
4. C	12. A	20. D	
5. C	13. A	21. A	
6 E	14. C	22. A	
7. E	15. A	23. B	
8 B	16. B	24. C	