ACTM State Mathematics Contest - Calculus

April 24, 2010

1. $\lim_{x\to 0}\left[1+2x^3+\sqrt{9+5x}+\sin(\ln(x+e^x))\right]=$

   A. -2

   B. 0

   C. 5

   D. Does not exist

   E. None of the above.

2. $\lim_{x\to 0} x^2\cos\dfrac{1}{x}=$

   A. -1

   B. 0

   C. 1

   D. ∞

   E. None of the above.

3. $\lim_{x\to 0}\dfrac{\sin(2x)}{x^2+x}=$

   A. 0

   B. 1

   C. 2

   D. Does not exist

   E. None of the above.

4. The limit of the Riemann sum $\lim_{n\to\infty}\sum_{k=1}^{n}\dfrac{(k-1)^4}{n^5}$

   A. $1/5$

   B. $1/4$

   C. $4/5$

   D. $5/4$

   E. None of the above.

5. Let

$$h(x) = \begin{cases} x^2 + 1 & \text{if } x < 0, \\ 2x + 1 & \text{if } x \geq 0. \end{cases}$$

Then

A. $\lim_{x \to 0^-} h(x) = 2$

B. $\lim_{x \to 0^+} h(x) = 0$

C. $h(x)$ is continuous at 0

D. $h(x)$ is differentiable at 0

E. None of the above

6. Let

$$h(x) = \begin{cases} 2 & \text{if } x < 2, \\ a\cos(x-2) & \text{if } x \geq 2. \end{cases}$$

Then $h(x)$ is continuous at 2 if $a =$

A. 0

B. 1/2

C. 1

D. 2

E. None of the above.

7. All the non removable points of discontinuity of the function $f(x) = \frac{x}{\sin x}$ are

A. $n\pi, n = \pm 1, \pm 2, \cdots$

B. $n\pi, n = 1, 2, \cdots$

C. $2n\pi, n = 0, \pm 1, \pm 2, \cdots$

D. $n\pi, n = 0, \pm 1, \pm 2, \cdots$

E. None of the above.

8. $f(x) = |x|$

A. is differentiable at 0

B. is continuous at 0

C. has a tangent line at 0 with the slope 1

D. has a tangent line at 0 with the slope -1

E. None of the above.

9. Let

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

A. $f(x)$ is differentiable at 0
B. $\lim_{x \to 0} f(x) = 1$
C. $f(x)$ is continuous at 0
D. $\lim_{x \to 0} f(x) = \infty$
E. None of the above.

10. Let $f(1) = 2, f'(1) = 1, g(1) = 3, g'(1) = 4$ and $h(x) = f(x)g(x)$. Then

A. $h'(1) = 4$
B. $h'(1) = 6$
C. $h'(1) = 11$
D. $h'(1) = 24$
E. None of the above.

11. If f is differentiable, then $\frac{d}{dx}[f(x^2)] =$

A. $f'(2x)$
B. $2f'(x^2)$
C. $2xf'(x^2)$
D. $2xf'(2x)$
E. None of the above.

12. The function $f(x) = x^3 + 4x + 10000$ has

A. a unique zero
B. two zeros
C. three zeros
D. no zeros
E. None of the above.

13. If $y = \tan(\sin(x^2))$, then $y' =$

A. $2x \sec^2(x) \cos(x^2)$
B. $\sec^2(\sin(x^2)) \cos(x^2)$
C. $2x \sec^2(x) \cos(x)$
D. $2x \sec^2(\sin(x^2)) \cos(x^2)$
E. None of the above.

14. If $xy^2 + e^x \sin y = 1$ (y is a function of x), then $y' =$

    A. $-\frac{e^x \sin y}{2xy + e^x \cos y}$

    B. $-\frac{y^2 + e^x \sin y}{2xy + e^x \cos y}$

    C. $-\frac{y^2 + e^x \sin y}{e^x \cos y}$

    D. $\frac{1 - y^2 - e^x \sin y}{2xy + e^x \cos y}$

    E. None of the above.

15. If $y = x^x$, then $y' =$

    A. x^{x-1}

    B. x^x

    C. $x^x(1 + \ln x)$

    D. $x^x \ln x$

    E. None of the above.

16. The equation of the tangent line to $f(x) = e^x$ that is parallel to the line $y = x + 2$ is

    A. $y = x + 1$

    B. $y = x$

    C. $y = xe^x$

    D. $y = xe^x + 1$

    E. None of the above.

17. If $f(x) + x^2[f(x)]^3 - 10 = 0$ and $f(1) = 2$, find $f'(1)$.

    A. $-16/13$

    B. -24

    C. -12

    D. $-13/16$

    E. None of the above.

18. Let $f(x) = \int_{1/x}^{x} \sin(t^2) dt$, then $f'(1) =$

    A. 0

    B. $\sin 1$

    C. $2 \sin 1$

    D. $\frac{3}{2} \sin 1$

    E. None of the above.

19. $\int e^{\tan^{-1} x}(\frac{1+x+x^2}{1+x^2})dx =$

A. $\frac{xe^{\tan^{-1} x}}{1+x^2} + c$

B. $(x^2+1)e^{\tan^{-1} x} + c$

C. $\frac{e^{\tan^{-1} x}}{1+x^2} + c$

D. $xe^{\tan^{-1} x} + c$

E. None of the above

20. $\int_{-1}^{1} \frac{d}{dx}(\tan^{-1}(\frac{1}{x}))dx.$

A. $\frac{-\pi}{2}$

B. $\frac{-\pi}{4}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

E. None of the above.

21. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

A. 10

B. 16

C. 18

D. 36

E. None of the above.

22. In a certain region the temperature (in ^oF) t hours after 9 A.M. was modeled by the function $T(t) = 50 + 14\sin\frac{\pi t}{12}$. Find the average temperature, approximated to one decimal place, during the period from 9 A.M. to 9 P.M.

A. 56.1^oF

B. 68.1^0F

C. 58.9^oF

D. 64.2^oF

E. None of the above.

23. An equation of the non-horizontal tangent line to the curve $y = x^2$ that passes through the point $(2, 0)$ is given by

A. $y = 4x - 8$

B. $y = 8x - 16$

C. $y = -4x + 8$

D. $y = -8x + 16$

E. None of the above.

24. The function $y = \frac{x}{1+x^2}$ decreases on

    A. $(-1, 1)$
    B. $(-1, \infty)$
    C. $(1, \infty)$
    D. $(-\infty, \infty)$
    E. None of the above.

25. An open water tank with a square base and rectangular sides is to be constructed. The cost of painting the inside of the tank will be minimum if the tank's

    A. depth is half of its width
    B. depth is a quarter of its width
    C. depth is 3/4 of its width
    D. depth is the same as its width
    E. None of the above.

(Tie-Breaker 1) Two drivers started from the same place in Conway at 4:30 pm and arrived at the same place in Little Rock at 5:00 pm. Prove that two drivers were driving at exactly the same speed at the exact same time between 4:30 pm and 5:00 pm.

(Tie-Breaker 2)Prove that

$$\ln(1+x) < x, \quad x > 0.$$

(Tie-Breaker 3) Find the shortest distance between the parabola $y^2 = 4x$ and the circle $x^2 + y^2 - 24y + 128 = 0$.