State Calculus Contest

April 2008

1) $\lim_{x\to 0} x\cot 2x$

(a) 0 (b) 1/2 (c) 1 (d) 2 (e) none of these

2) $\frac{d}{dx}\int_{-3}^{1} e^{\pi}\,dt =$

(a) 0 (b) e (c) π (d) e^{π} (e) none of these

3) If $f(1) = 2,\ g(1) = 1,\ f'(1) = 5,\ g'(1) = -2,$ then $\left(\frac{f}{f+g}\right)'(1)$ is

(a) 0 (b) 1 (c) 5/3 (d) 2/3 (e) none of these

4) $\int x^3\sqrt{1+x^2}\,dx =$

(a) $\frac{x^4\sqrt{1+x^2}}{4}+c$ (b) $\frac{x^4}{4}\sqrt{1+\frac{x^3}{3}}+c$ (c) $\frac{\left(1+x^2\right)^{3/2}\left(3x^2-2\right)}{15}+c$

(d) $x^3\sqrt{1+\frac{x^3}{3}}+c$ (e) none of these

5) If $f(x) = \begin{cases} x & x<0 \\ x^2+1, & 0\le x\le 2 \\ 1-x, & x>2 \end{cases}$, then $\lim_{x\to 2^-} f(x)$

(a) -1 (b) 0 (c) 1 (d) 5 (e) none of these

6) $\frac{d}{dx}\log_2\left(x^2 + \sin x\right)$

(a) $\frac{2x + \cos x}{\ln 2\left(x^2 + \sin x\right)}$ (b) $\frac{2x + \cos x}{x^2 + \sin x}$ (c) $\frac{\ln 2\left(2x + \cos x\right)}{x^2 + \sin x}$

(d) $2x + \cos x$ (e) none of these

7) If $f(x) = \frac{2}{x}$ and $g(x) = \frac{x}{x^2 + 1}$, then $(f \circ g)'(x) =$

(a) $\frac{2\left(x^2 - 1\right)}{x^2\left(x^2 + 1\right)^2}$ (b) $-\frac{2\left(3x^2 + 1\right)}{x^2}$ (c) $-\frac{1}{x^2}$ (d) $\frac{2\left(x^2 - 1\right)}{x^2}$ (e) none of these

8) The function $f(x) = x^4 - 12x^3 + 48x^2 - 64x$ is concave upwards on the interval(s)

(a) $x < 2$ and $x > 4$ (b) $2 < x < 4$ (c) $1 < x < 4$ and $x > 4$

(d) $x < 1$ (e) none of these

9) If $f(x) = \int_{-x}^{2x} \sqrt[3]{1 + t^2}\,dt$ then $f'(x)$ is

(a) $2\sqrt[3]{1 + 4x^2} + \sqrt[3]{1 + x^2}$ (b) $\sqrt[3]{1 + 4x^2} + \sqrt[3]{1 + x^2}$

(c) $2\sqrt[3]{1 + 4x^2} - \sqrt[3]{1 + x^2}$ (d) $2\sqrt[3]{1 + 4x^2} - \sqrt[3]{1 + x^2}$

(e) none of these

10) If $y = \left|x^2 - 1\right|$, then y' is

(a) $2x$ (b) $\frac{1}{\left|x^2 - 1\right|}$ (c) $2x\left|x^2 - 1\right|$ (d) $\frac{2x}{\left|x^2 - 1\right|}$ (e) none of these

11) The values of a and b such that $\lim_{x \to 0} \frac{\sqrt{ax + b} - 2}{x} = 1$ is

(a) $a = 1, b = 0,$ (b) $a = 0, b = 4,$ (c) $a = 2, b = 1,$ (d) $a = 4, b = 4,$

(e) none of these

12) If $x^n + y^n = 1$, for some nonzero real number n, then y'' is

(a) $-\dfrac{x^{n-2}}{y^{n-2}}$ (b) $-\dfrac{x^{n-2}}{y^{2n-1}}$ (c) $-(n-1)\dfrac{x^{n-2}}{y^{2n-1}}$ (d) $\left(1-x^n\right)^{1/n}$

(e) none of these

13) A water tank has the shape of an inverted cone with base radius of 2*m* and height of 4*m*. If water is being pumped into the tank at a rate of $2m^3$/min, at what rate is the water level rising when the tank is 3*m* deep?

(a) $\dfrac{8}{9\pi}$ *m*/min (b) $\dfrac{8\pi}{9}$ *m*/min (c) $\dfrac{9}{8\pi}$ *m*/min (d) $\dfrac{9\pi}{8}$ *m*/min (e) none of these

14) The absolute maximum and minimum values of $x^2 - xy + y^2 = 3$ are?

(a) $1, -1$ (b) $\sqrt{3}, -\sqrt{3}$ (c) $2, -2$ (d) $3, -3$ (e) none of these

15) The values of a and b such that the function $f(x) = x^3 + ax^2 + bx + 1$ has a local maximum at $x = -3$ and a local minimum at $x = -1$ are

(a) $a = -3, b = -9$ (b) $a = -6, b = 9$ (c) $a = 3, b = -9$ (d) $a = 6, b = 9$

(e) none of these

16) If *f* and g are differentiable functions such that $f(g(x)) = x$ and $f'(x) = 1 + f^2(x)$ then $g'(x)$ is

(a) x (b) $\dfrac{1}{1+x^2}$ (c) $1 + x^2$ (d) 1 (e) none of these

17) The equation of the tangent to $\tan^{-1}(x + y - 1) = x^2 + y^2$ at $(1,0)$ is

(a) $y = x + 1$ (b) $y = 2x + 1$ (c) $y = 2x - 2$

(d) $y = 7x - 7$ (e) none of these

18) If $f(x) = \int_1^x \left(\int_{t^2}^1 \sqrt{1 + r^4}\,dr \right) dt$, then $f''(x)$ is

(a) $\sqrt{1 + x^4}$ (b) $2x\sqrt{1 + x^4}$ (c) $2x\sqrt{1 + x^8}$ (d) $\dfrac{2x^2\left(x^4 + 3\right)}{\left(1 + x^4\right)^{3/2}}$

(e) none of these

19) A piece of string is 10 ft long and is cut into 2 pieces. One piece is used to form a square while the other is used to form a circle. How much should be used for the circle such that the sum of their areas is a maximum.

(a) $10\,ft$ (b) $0\,ft$ (c) $5\,ft$

(d) $\dfrac{5}{\pi + 4}\,ft$ (e) none of these

20) If a and b are positive real numbers, the area between $y = ax^2$ and $y = mx - ax^2$ is

(a) $\dfrac{b^3}{24a^2}$ (b) $\dfrac{b^3}{8a^2}$ (c) $\dfrac{b^3}{4a^2}$ (d) $\dfrac{b^3}{2a^2}$ (e) none of these

21) The value of $\sum_{i=3}^{n-1} i^2 + 1$ is

(a) $\dfrac{(n-2)\left(2n^2 + 7n + 2\right)}{6}$ (b) $\dfrac{(n-1)\left(2n^2 + 5n + 12\right)}{6}$ (c) $\dfrac{n\left(2n^2 + 3n + 7\right)}{6}$

(d) $\dfrac{(n-3)\left(2n^2 + 3n + 16\right)}{6}$ (e) none of these

22) If $f(x)$ is differentiable at $x = a \neq 0$ then $\lim_{x \to a} \dfrac{f\left(x^2\right) - f\left(a^2\right)}{x^2 - a^2}$ is

(a) $f'(a)$ (b) $f(a)\,f'(a)$ (c) $f'\left(a^2\right)$ (d) $\dfrac{f'\left(a^2\right)}{a^2}$ (e) none of these

23) The value of $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + \left(\frac{i}{n}\right)^2}$ is

(a) 1 (b) $\frac{\pi}{8}$ (c) $\frac{\pi}{2}$ (d) π (e) none of these

24) If $f(x)$ is differentiable and

$$f(1) = f'(1) = 1, \; f(2) = f'(2) = 2, \; f(3) = f'(3) = 3,$$

and

$$g(x) = f\left(x + f\left(x + f(x)\right)\right),$$

then $g'(1)$ is

(a) 3 (b) 9 (c) 12 (d) 15 (e) none of these

25) If the vertical motion of a spring is given by $y = e^{-t} \sin\left(\sqrt{3}\, t\right)$, then the times at which the springs position is at a maximum is when (n is a positive integer)

(a) $\frac{1}{\sqrt{3}}\left(\frac{\pi}{3} + n\pi\right)$ (b) $\frac{1}{\sqrt{3}}\left(\frac{\pi}{3} + 2n\pi\right)$ (c) $\frac{\pi}{3} + n\pi$

(d) $\frac{\pi}{3} + 2n\pi$ (e) none of these

Tie Breakers

1) If $f(x)$ is differentiable, find all functions that satisfy $2\int_0^x t\, f(t)\,dt + f(x) + 3 = 0$ for all x.

2) If $f(x)$ is differentiable at $x = 0$ where $f(0) = 0$ and $f(x) \leq 0$ for all x, find $f'(0)$.

3) Evaluate

$$\int_0^{\pi} \frac{x \sin x\, dx}{1 + \cos^2 x}.$$

State Calculus Contest 2008 – Solutions

1 – b	11 – d	21 – d
2 – a	12 – c	22 – c
3 – b	13 – a	23 – e
4 – c	14 – c	24 – d
5 – d	15 – d	25 – b
6 – a	16 – b	
7 – d	17 – e	
8 – a	18 – e	
9 – a	19 – a	
10 – e	20 – a	

Tie Breaker 1

If we differentiate

$$2\int_0^x t\, f(t)\, dt + f(x) + 3 = 0,$$

we obtain

$$2x\, f(x) + f'(x) = 0,$$

which integrates giving $f(x) = c\,e^{-x^2}$. Back-substituting into the original equation gives

$$2c\int_0^x t\,e^{-t^2}\, dt + c\,e^{-x^2} + 3 = 0.$$

from which we obtain, after integrating, $c = -3$ giving the solution as $f(x) = -3e^{-x^2}$.

Tie Breaker 2

Since *f*(*x*) is differentiable at $x = 0$ and $f(0) = 0,$ using the definition of a derivative at a point then

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{f(x)}{x} = f'(0).$$

Since $f(x) \le 0,$ for all *x*, then $\frac{f(x)}{x} \ge 0,$ if $x < 0$ and $\frac{f(x)}{x} \le 0,$ if $x > 0$ from which it follows that $\lim_{x\to 0} \frac{f(x)}{x} = 0$ giving $f'(0) = 0.$

Tie Breaker 3

If we denote the integral by I, then

$$I = \int_0^\pi \frac{x \sin x \, dx}{1 + \cos^2 x}.$$

Making the substitution $x = \pi - u$ gives

$$I = -\int_\pi^0 \frac{(\pi - u)\sin(\pi - u)\,du}{1 + \cos^2(\pi - u)} = \int_0^\pi \frac{(\pi - u)\sin u \, du}{1 + \cos^2 u} = \pi \int_0^\pi \frac{\sin u \, du}{1 + \cos^2 u} - I.,$$

Thus,

$$2I = \pi \int_0^\pi \frac{\sin u \, du}{1 + \cos^2 u} = -\pi \tan^{-1} \cos u \Big|_0^\pi = 2\pi \tan^{-1} 1 = \frac{\pi}{2},$$

from which we find that $I = \dfrac{\pi}{4}$.